Seminar: Group rings and dimensions

F. Hofmann/C. Löh/M. Wasmeier January 2026

Group rings combine the multiplicative structure of a group with a base ring. They appear naturally in many algebraic and topological situations. Despite their simple definition, even basic questions about group rings (e.g., on the existence of non-trivial zero divisors) easily lead to surprising challenges.

In this seminar, we will study elementary algebraic properties of group rings and learn about embeddings of group rings into rings with better algebraic properties. In particular, we will discuss the Kaplansky conjectures, the von Neumann dimension, and the Atiyah conjecture.

Literature. The seminar is based on several sources, these are explicitly indicated for each talk. However, you may and should also consult other sources.

Prerequesites. Most of the talks will be accessible with basic knowledge on groups, rings, and modules. Some of the talks benefit from experience in analysis, topology or homological algebra.

Admin and preparation. Please take the general advice on seminars into account: https://loeh.app.uni-regensburg.de/teaching/seminar_preparation.pdf

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Groups and group rings

Talk 1 (Free groups and presentations)**.**
Literature: [CC10, Appendix D] [Löh17, Chapter 2.2.2–2.2.4, 3.3.1]

- Define free groups via their universal property, give examples and non-examples.
- Construct free groups using reduced words.
- Define the rank of a free group.
- Introduce group presentations, define finitely generated and finitely presented groups and give examples.

Talk 2 (Group rings)**.**
Literature: [Hal54, Theorem 1] [MS02, Chapter 3.1, 3.2] [Put]

- Introduce group rings and discuss their universal property.
- Give explicit examples for group rings.
- Discuss ring theoretic properties of group rings, in particular being commutative and being Noetherian.

Talk 3 (von Neumann regular rings).
Literature: [Aus57] [Goo79, Chapter 1] [Lüc02, Chapter 8.2.2]

- Define von Neumann regular rings and give examples. (Caution: Don't confuse them with regular rings!)
- Introduce projective modules.
- Present equivalent characterizations of von Neumann regularity [Lüc02, Lemma 8.18]. Proof the equivalence for one characterization of your choice.
- Show that every element in a von Neumann regular ring is either a zero-divisor or a unit.
- Explain what is known about von Neumann regularity of group rings.

Talk 4 (The Kaplansky conjectures).
Literature: [Gar24, Chapter 1.1, 1.2] [Pas77, Chapter 13.1, 13.2]

- Introduce the idempotent conjecture, the zero divisor conjecture and the unit conjecture.
- Show that the unit conjecture implies the zero divisor conjecture and that the zero divisor conjecture implies the unit conjecture.
- Introduce the unique product property and give examples for groups with this property.
- Show that group rings of groups with the unique product property satisfy the unit conjecture.

Talk 5 (A counterexample for the unit conjecture).
Literature: [Gar21] [Gar24, Chapter 1.6] [Pel23, Chapter 3]

- Recall the Kaplansky conjectures.
- Introduce the Promislow group P and motivate why it is a candidate for a counterexample to the unit conjecture.
- Prove that the Promislow group is torsion-free.
- Give a sketch of the proof that $\mathbb{F}_2P$ is a counterexample to the unit conjecture. (You don't have to do the explicit calculations!)

The von Neumann dimension

Talk 6 (The group von Neumann algebra).
Literature: [Kam19, Chapter 2.1, 2.2] [Lüc02, Chapter 1.1.1]

- Introduce the group von Neumann algebra $\mathcal{N}(G)$ as a completion of the complex group ring.
- Show that $\mathcal{N}(G)$ can also be described as the algebra of G-equivariant, bounded operators $\ell^2(G) \to \ell^2(G)$ [Kam19, Thm. 2.24].
- Discuss the examples $\mathcal{N}(G)$ for finite G and $\mathcal{N}(\mathbb{Z}^n)$.
- Optional: Define von Neumann algebras and state that group von Neumann algebras are examples for von Neumann algebras.

Talk 7 (The von Neumann dimension).
Literature: [Lüc98] [Lüc02, Chapter 6.1]
Note: We want to talk about the von Neumann dimension in the sense of [Lüc02, Definition 6.20], not [Lüc02, Definition 1.10].

- Introduce the von Neumann trace and the von Neumann dimension for finitely generated projective $\mathcal{N}(G)$-modules.
- Motivate why the dimension is defined as trace of a projection.
- Explain how to expand dimension functions with certain assumptions and define the von Neumann dimension for general $\mathcal{N}(G)$-modules.
- Discuss properties of the extended dimension function, in particular the extension property, additivity, cofinality and continuity.
- Calculate explicit examples.

Talk 8 (Hilbert $\mathcal{N}(G)$-modules).
Literature: [Lüc02, Chapter 1.1.2, 1.1.4, 6.2]

- Define Hilbert $\mathcal{N}(G)$-modules and give examples.
- Introduce Hilbert chain complexes and L^2-homology.
- Show that L^2-homology can be computed in terms of kernels of Laplacians.
- Explain the connection between finitely generated Hilbert modules and finitely generated projective modules over the group von Neumann algebra $\mathcal{N}(G)$.

Talk 9 (ℓ^2-Betti numbers).
Literature: [Kam19, Chapter 4.2] [Löh19, Chapter 2.1.3]
Note: This talk requires some familiarity with algebraic topology.

- Recall the von Neumann dimension and introduce ℓ^2-Betti numbers of G-spaces.
- Introduce the classifying space EG and ℓ^2-Betti numbers of groups.
- Calculate the 0-th ℓ^2-Betti number of a group.
- Explain how ℓ^2-Betti numbers behave for products, free products and finite index subgroups.
- Mention the connection between ℓ^2-Betti numbers and the Euler characteristic.
- Calculate the ℓ^2-Betti numbers of $\mathbb{Z}$.

The Atiyah conjecture

Talk 10 (The Atiyah conjecture).
Literature: [Aus13] [Gra14, Section 6] [Lüc02, Chapter 10.1] [Rei98, Chapter 5.1]

- Introduce the Atiyah conjecture, the strong Atiyah conjecture and the Atiyah conjecture in terms of modules [Lüc02, Conjecture 10.2, 10.3, Lemma 10.7].
- Explain the connection between the Atiyah conjecture and ℓ^2-Betti numbers.
- Show that the Atiyah conjecture implies the Kaplansky conjecture.
- State Linnell's theorem.
- Give a counterexample for the strong Atiyah conjecture. Shortly mention the results by Austin and Grabowski.

Talk 11 (Noncommutative localization).
Literature: [Lüc02, Chaper 8.2.1, 10.2.2] [Rei98, Appendix III]

- Recall the concept of localization from commutative algebra.
- Explain Ore localization and, more generally, universal localization.
- Introduce the division closure and the rational closure of a ring extension.
- Show that a von Neumann regular ring is division closed and rationally closed in every overring.
- Show factorization results for ring homomorphisms out of universal localizations [Rei98, Proposition 13.17].

Talk 12 (K_0 of rings).
Literature: [Ros94, Chapter 1.1] [Wei13, Chapter I.2, II.1, II.2]

- Explain the concept of group completion of a monoid.
- Recall the notion of projective modules and introduce K_0 of a ring.
- Explain why we only use *finitely generated* projective modules in the definition of K_0.
- Show that K_0 is a functor from the category of rings to the category of abelian groups.
- Calculate $K_0(R)$ for R a (skew-)field and for R a PID.
- Show that $[P] = [Q] \in K_0(R)$ if and only if P and Q are stably isomorphic.

Talk 13 (The strong Atiyah conjecture for torsion-free groups).
Literature: [Rei98, Chapter 5.2, 8.4] [Lüc02, Chapter 10.1, 10.2]

- Recall the strong Atiyah conjecture.
- Explain a general strategy to prove the strong Atiyah conjecture [Lüc02, Lemma 10.26].

- Prove that a torsion-free group G satisfies the strong Atiyah conjecture if and only if $\mathcal{D}(G)$ is a skew field [Rei98, Proposition 8.30] [Lüc02, Lemma 10.39].
- State and explain Linnell's theorem.

Possible further talks:

- ℓ^2-Betti numbers for topological spaces
- Connections to amenability

References

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