Seminar: Growth of Groups

F. Hofmann/C. Löh/M. Wasmeier July 2026

No reasonable classification of finitely generated groups is possible. One attempt at structuring the universe of finitely generated groups is to sort groups by their growth types. Examples of notions of growth of groups include the following:

- Word-length growth measures the growth of balls in groups of growing radius with respect to the word-length;
- subgroup growth counts the number of subgroups of growing (finite) index;
- residual finiteness growth measures how difficult it is to detect non-triviality of elements by passing to finite quotients.

The growth of fundamental groups (a notion from algebraic topology) of manifolds leads to interesting group-theoretic applications in topology and geometry. Even though the above notions of growth are substantially different and not directly related, many intriguing parallels exist. For instance, the question of which classes of groups have polynomial word-length growth or polynomial subgroup growth leads to similar answers (and even, to a certain extent, to similar proof techniques).

In this seminar, we will study word-length growth, subgroup growth, and residual finiteness growth. We will illustrate these notions in concrete examples and learn how to approach the polynomial growth theorems. Moreover, we will discuss some basic applications in topology and geometry.

Literature. The seminar is based on several sources, these are explicitly indicated for each talk. However, you may and should also consult other sources.

Prerequesites. Most of the talks will be accessible with basic knowledge on groups, rings, and modules. Some of the talks benefit from experience in topology or geometry.

Admin and preparation. Please take the general advice on seminars into account: https://loeh.app.uni-regensburg.de/teaching/seminar_preparation.pdf

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Word-length growth

Talk 1 (Finitely generated and finitely presented groups)**.**
Literature: [DK18, Chapter 7.1–7.4], [dlH00, Chapter III.A, V.A], [CSC10, Appendix D]

- Define free groups, give examples and non-examples.
- Construct free groups via equivalence classes of words or reduced words, mention the other possibility.
- Define the rank of free groups.
- Introduce group presentations, define finitely generated and finitely presented groups and give examples.

Talk 2 (Cayley graphs and growth functions)**.**
Literature: [dlH00, Chapter IV.A, VII.C.21, 22], [CSC10, Chapter 6.2–6.4],

- Introduce graphs and Cayley graphs and give examples of Cayley graphs.
- Define growth functions of finitely generated groups and give examples (e.g. free, free abelian, finite groups).
- Compute a growth function of the Heisenberg group. Why is it interesting?

Talk 3 (Quasi-isometries)**.**
Literature: [CSC10, Chapter 6.4, 6.13], [dlH00, Chpater IV.B]

- Define quasi-isometries of metric spaces.
- Introduce quasi-equivalence of growth functions.
- Define the growth type of a finitely generated group and prove that it is independent of the finite generating set.
- Give examples for groups with polynomial growth and exponential growth.

Talk 4 (Groups of polynomial and intermediate growth)**.**
Literature: [CSC10, Chapter 6.8–6.9], [dlH00, Chapter VIII.B]

- Define (virtually) nilpotent groups, give examples and non-examples.
- Prove that finitely generated nilpotent groups have polynomial growth.
- State Gromov's polynomial growth theorem.
- Mention the Grigorchuk group as an example for a group with intermediate growth.

Talk 5 (Growth of solvable groups)**.**
Literature: [DK18, Chapter 14.3–14.4], [Mil68], [CSC10, Chapter 6.7]

- Define solvable groups, give examples and non-examples.
- Show that nilpotent groups are solvable.
- Prove Milnor's and Wolf's theorem.
- Give an example of a solvable group with exponential growth.

Talk 6 (Gromov's polynomial growth theorem)**.**
Literature: [DK18, Chapter 16.6, Proposition 14.39], [vdDW84]

- Introduce ultra limits and asymptotic cones.
- Prove [DK18, Proposition 14.39] about group extensions and subexponential growth.
- Prove Gromov's polynomial growth theorem. Sketch the part about asymptotic cones, focus on the induction argument.

Subgroup growth

Talk 7 (Residually finite groups)**.**
Literature: [DK18, Chapter 7.11], [CSC10, Chapter 2.1–2.3, 2.6], [LS03, Window 4], [Wil24, Chapter 3]

- Define residually finite groups, give examples and non-examples.
- Prove some inheritance properties.
- Define profinite groups.
- Define the profinite completion of a group and show that it is a profinite group, mention the topology on profinite completions.

Talk 8 (Subgroup growth)**.**
Literature: [LS03, Chapter 0, 1.1, 1.2, 1.4, 1.11]

- Define the (generating) rank of a group and the subgroup growth function.
- Prove inheritance properties and give examples.
- Prove that nilpotent and solvable groups of finite rank have polynomial subgroup growth.
- Prove that free groups have superexponential subgroup growth.

Talk 9 (Linear groups with PSG)**.**
Literature: [LS03, Chapter 5.2, Window 7, 9, 10]

- Briefly introduce algebraic groups and discuss the examples GL_n and SL_n.
- State Lubotzky's Theorem.
- Explain the prime number theorem.
- Prove that every finitely generated linear group of characteristic zero with wPSG is virtually solvable.

Talk 10 (The prosolvable case)**.**
Literature: [LS03, Chapter 5.4], [Isa09, Chapter 8.B–C]

- Define chief factors and give examples.
- Define the p-core $O_p(G)$ for a finite group G and a prime p and give examples.
- Prove Hall's theorem [Isa09, Chapter 8.C]
- Show that every group of prosolvable type with weak PSG has finite upper rank [LS03, Chapter 5.4].

Talk 11 (Finite simple groups and upper chief factors)**.**
Literature: [LS03, Chapter 5.3, Window 2], [His11]

- Explain groups of Lie type and give examples.
- State the CFSG (Classification of finite simple groups).
- Define upper chief factors and give examples.
- Show that every wPSG group has restricted upper chief factors.
- Mention the statement of [LS03, Proposition 5.3.6].

Talk 12 (Polynomial subgroup growth theorem)**.**
Literature: [LS03, Introduction of Chapter 5, Chapter 5.5]

- Prove that finitely generated residually finite groups have finite upper rank iff they are virtually solvable of finite rank [LS03, Theorem 5.2].
- Combine the results of the last talks to prove the PSG-Theorem [LS03, Introduction of Chapter 5].

Residual finiteness growth

Talk 13 (Residual finiteness growth)**.**
Literature: [BR10, Introduction, Chapter 3, 5]

- Define residual finiteness growth.
- Prove inheritance properties.
- Give examples of residual finiteness growth functions.
- Give a characterization of finitely generated nilpotent groups in terms of finite nilpotent quotients [BR10, Theorem 0.3].

Talk 14 (The Grigorchuk group)**.**
Literature: [dlH00, Chapter VIII.B, VIII.F], [BR10, Section 4], [BSP20]

- Define the Grigorchuk group.
- Prove that it has intermediate word-length growth.
- Mention that its subgroup growth type is $n^{\log(n)}$.
- Prove that its residual finiteness growth is at most exponential.

References

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[LS03] Alexander Lubotzky and Dan Segal. *Subgroup growth*, volume 212 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2003.

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