Algebraic Topology I – Exercises

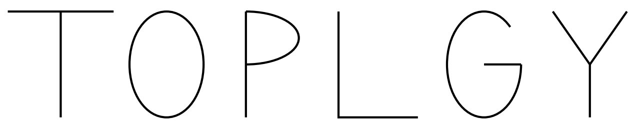
Prof. Dr. C. Löh — Sheet 0, October 16, 2015

Exercise 1 (very small categories)**.**

1. Which algebraic concept corresponds to categories with exactly one object?
2. Which algebraic concept corresponds to functors between categories with exactly one object?
3. Prove or disprove: In a category with exactly one object and exactly 2015 morphisms, every morphism is an isomorphism.
4. Prove or disprove: In a category with exactly two objects and exactly 2015 morphisms, every morphism is an isomorphism.

Exercise 2 (TOPOLOGY)**.**

1. Recall the definition of *connectedness* and *path-connectedness* of topological spaces.
2. Classify the following six subspaces of $\mathbb{R}^2$ up to homeomorphism and prove this classification result.



Exercise 3 (balls, spheres, simplices)**.** Let $n \in \mathbb{N}$ and let $S^{-1} := \emptyset$.

1. Prove that $(\Delta^n, \partial\Delta^n)$ and (D^n, S^{n-1}) are homeomorphic as pairs of spaces.
2. Prove that D^n/S^{n-1} is homeoomorphic to S^n.

   Hints. Let X be a topological space and let $A \subset X$. We then write X/A for the quotient space $X/\sim$, where "$\sim$" is the equivalence relation

   $$\bigl\{(a,b) \bigm| a,b \in A\bigr\} \cup \bigl\{(x,x) \bigm| x \in X\bigr\} \subset X \times X$$

   on X.
3. Let $N := (0,\dots,0,1) \in S^n$. The map

   $$\begin{aligned} s_n\colon S^n \setminus \{N\} &\longrightarrow \mathbb{R}^n \\ (x_1,\dots,x_{n+1}) &\longmapsto \frac{1}{1-x_{n+1}} \cdot (x_1,\dots,x_n) \end{aligned}$$

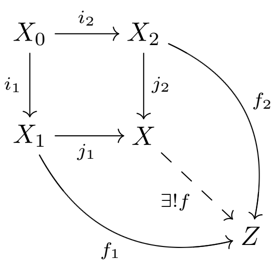
   is called *stereographic projection*. Give a geometric interpretation of this map and prove that it is a homeomorphism.

Illustrate your arguments graphically!

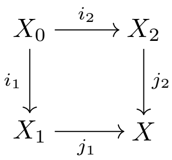
Please turn over

Exercise 4 (universal property of pushouts)**.** Let X_0, X_1, X_2 be topological spaces and let $i_1 \colon X_0 \longrightarrow X_1$, $i_2 \colon X_0 \longrightarrow X_2$ be continuous maps. A topological space X together with continuous maps $j_1 \colon X_1 \longrightarrow X$, $j_2 \colon X_2 \longrightarrow X$ *satisfies the universal property of this pushout* if the following holds: We have $j_2 \circ i_2 = j_1 \circ i_1$ and for all topological spaces Z and all continuous maps $f_1 \colon X_1 \longrightarrow Z$, $f_2 \colon X_2 \longrightarrow Z$ with $f_1 \circ i_1 = f_2 \circ i_2$ there exists exactly one continuous map $f \colon X \longrightarrow Z$ with

$$f \circ j_1 = f_1 \quad \text{and} \quad f \circ j_2 = f_2.$$



In this case, we also say that



is a *pushout diagram of topological spaces.*

1. Prove that the pushout $X := X_1 \cup_{X_0} X_2$ of i_1 and i_2, together with the continuous maps $j_1 \colon X_1 \longrightarrow X$, $j_2 \colon X_2 \longrightarrow X$ induced by the canonical inclusions of X_1 and X_2 into $X_1 \sqcup X_2$, satisfies the universal property of this pushout.

2. Let X' be a topological space that, together with the continuous maps $j_1' \colon X_1 \longrightarrow X'$, $j_2' \colon X_2 \longrightarrow X'$, satisfies the universal property of this pushout. Prove that there is a canonical homeomorphism $X \cong X'$.

Bonus Problem (Peano curves)**.** Show that there exist surjective continuous maps $[0,1] \longrightarrow [0,1] \times [0,1]$. Can such a map be injective?

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no submission;
these exercises will be discussed in the exercise class on October 21, 2015