Algebraic Topology I – Exercises

Prof. Dr. C. Löh — Sheet 1, October 16, 2015

Exercise 1 (isomorphisms of pairs of spaces)**.** Let (X, A) and (Y, B) be pairs of spaces. Prove or disprove:

1. If $(X, A) \cong (Y, B)$, then $X \cong Y$ and $A \cong B$.
2. If $A \cong B$ and $X \cong Y$, then $(X, A) \cong (Y, B)$.
3. If $(X, A) \cong (Y, B)$, then $X/A \cong Y/B$.
4. If $X/A \cong Y/B$, then $(X, A) \cong (Y, B)$.

Hints. Let X be a topological space and let $A \subset X$. We then write X/A for the quotient space $X/\sim$, where "$\sim$" is the equivalence relation

$$\bigl\{(a, b) \bigm| a, b \in A\bigr\} \cup \bigl\{(x, x) \bigm| x \in X\bigr\} \subset X \times X$$

on X.

Exercise 2 (sections/Schnitte)**.** Let C be a category, let $X, Y \in \operatorname{Ob}(C)$ and let $p \in \operatorname{Mor}_C(X, Y)$. The morphism p *admits a (right) section in* C if there exists a morphism $s \in \operatorname{Mor}_C(Y, X)$ satisfying $p \circ s = \operatorname{id}_Y$.

1. State and prove a characterisation of those morphisms in the category Set of Sets that admit a right section.
2. Does the analogous classification also hold in the categories $\mathsf{Vect}_{\mathbb{R}}$, Group, Top ?

Exercise 3 (constructing the real projective plane out of simple pieces)**.** Prove that there are pushout diagrams of topological spaces of the following type:

$$\begin{array}{ccc} S^0 & \longrightarrow & \mathbb{R}P^0 \\ \downarrow & & \downarrow \\ D^1 & \longrightarrow & \mathbb{R}P^1 \end{array} \qquad\qquad \begin{array}{ccc} S^1 & \longrightarrow & \mathbb{R}P^1 \\ \downarrow & & \downarrow \\ D^2 & \longrightarrow & \mathbb{R}P^2 \end{array}$$

In particular, describe all of the maps in the diagram explicitly and illustrate your arguments graphically.

Please turn over

Exercise 4 (morphisms in the simplex category)**.** For $n \in \mathbb{N}_{>0}$ and $j \in \{0, \dots, n\}$ we define

$$d^n_j \colon \Delta(n-1) \longrightarrow \Delta(n)$$
$$k \longmapsto \begin{cases} k & \text{if } k < j \\ k+1 & \text{if } k \geq j; \end{cases}$$

for $n \in \mathbb{N}$ and $j \in \{0, \dots, n\}$ we define

$$s^n_j \colon \Delta(n+1) \longrightarrow \Delta(n)$$
$$k \longmapsto \begin{cases} k & \text{if } k \leq j \\ k-1 & \text{if } k > j. \end{cases}$$

Clearly, all of these maps are morphisms in the simplex category Δ.

1. Prove that every morphism in Δ is a composition of finitely many of the morphisms above.
2. Let $n \in \mathbb{N}$.
   - Prove that for all $j, k \in \{0, \dots, n+1\}$ with $j < k$ we have
     $$d^{n+1}_k \circ d^n_j = d^{n+1}_j \circ d^n_{k-1}.$$
   - State and prove for $j \in \{0, \dots, n+1\}$, $k \in \{0, \dots, n\}$ an alternative representation for $s^n_k \circ d^{n+1}_j$.

Bonus Problem (triangulation/Triangulierungen)**.** A *simplicial complex* is a pair (V, S) consisting of a set V and a set S of finite subsets of V satisfying the following property: For all $\sigma \in S$ and all $\tau \subset \sigma$ we have $\tau \in S$. Elements of V are called *vertices*, elements of S are called *simplices*.

Let $K := (V, E)$ be a simplicial complex whose vertex set V is totally ordered by "$\leq$". If $\sigma = \{v_0, \dots, v_n\} \in S \setminus \{\emptyset\}$ with $v_0 < v_1 < \dots < v_n$, we write $\|\sigma\| := \Delta^n$; if $\tau = \{v_{j_0}, \dots, v_{j_m}\} \neq \emptyset$ and $j_0 < \dots < j_m \in \{0, \dots, n\}$, we consider the linear map $i_{\tau,\sigma} \colon \|\tau\| \longrightarrow \|\sigma\|$ defined on the unit vectors by $e_{k+1} \longmapsto e_{j_k+1}$ for all $k \in \{0, \dots, m\}$. The *geometric realisation of* K is the topological space

$$\|K\| := \Bigl(\bigsqcup_{\sigma \in S \setminus \{\emptyset\}} \|\sigma\| \Bigr) \Big/ {\sim},$$

where "$\sim$" is the equivalence relation on $\bigsqcup_{\sigma \in S \setminus \{\emptyset\}} \|\sigma\|$ generated by

$$\forall_{\sigma \in S \setminus \{\emptyset\}} \ \forall_{\tau \subset \sigma, \tau \neq \emptyset} \ \forall_{x \in \|\tau\|} \quad x \sim i_{\tau,\sigma}(x).$$

A *triangulation* of a topological space X is a pair (K, f), where K is a simplicial complex and $f \colon \|K\| \longrightarrow$ is a homeomorphism.

1. Illustrate the construction of geometric realisation at the simplicial complex
$$\bigl(\{1, \dots, 6\}, \emptyset, \{1\}, \dots, \{6\}, \{1,4\}, \{1,6\}, \{2,4\}, \{2,5\}, \{3,5\}, \{3,6\}, \{4,5\}, \{4,6\}, \{5,6\}, \{4,5,6\}\bigr)$$
2. Give a triangulation of the Möbius strip.
3. Does every topological space admit a triangulation?

Submission before October 23, 10:00, in the mailbox