Algebraic Topology I – Exercises

Prof. Dr. C. Löh Sheet 10, December 18, 2015

Exercise 1 (small singular cycles)**.** Let X be a topological space, let $k \in \mathbb{N}_{>0}$. Prove or disprove:

1. If $\sigma \in \operatorname{map}(\Delta^k, X) \subset C_k(X)$ is a singular cycle of X, then k is odd.
2. If $\sigma, \tau \in \operatorname{map}(\Delta^k, X)$ and $\sigma + \tau$ is a singular cycle of X, then k is even.

Exercise 2 (algebraic Euler characteristic/algebraische Euler-Charakteristik)**.** Let R be a ring with unit that admits a nice notion rk_R of rank for finitely generated R-modules (e.g., fields, principal ideal rings, ...). A chain complex $C \in \operatorname{Ob}({}_R\mathsf{Ch})$ is *finite* if for every $k \in \mathbb{Z}$ the R-module C_k is finitely generated and $\{k \in \mathbb{Z} \mid C_k \not\cong 0\}$ is finite. The *Euler characteristic* of a finite chain complex $C \in \operatorname{Ob}({}_R\mathsf{Ch})$ is defined by

$$\chi(C) := \sum_{k \in \mathbb{Z}} (-1)^k \cdot \operatorname{rk}_R C_k.$$

Show that

$$\chi(C) = \sum_{k \in \mathbb{Z}} (-1)^k \cdot \operatorname{rk}_R\bigl(H_k(C)\bigr)$$

and explain which properties of rk_R you used in your arguments.

Exercise 3 (simplicial sets out of groups)**.** Let $\Delta(\mathsf{Set})$ be the category of *simplicial sets*, i.e., the category of contravariant functors $\Delta \longrightarrow \mathsf{Set}$ with natural transformations as morphisms.

1. Construct a functor $B \colon \mathsf{Group} \longrightarrow \Delta(\mathsf{Set})$ with the following property: for all groups G and all $n \in \mathbb{N}$ we have
$$B(G)\bigl(\Delta(n)\bigr) = G^n.$$
2. Calculate $H_1\bigl(C(B(G))\bigr)$ for Abelian groups G. What happens if G is not Abelian?

Exercise 4 (singular homology in degree 1)**.** Let X be a path-connected non-empty topological space.

1. Let $\alpha \in H_1(X;\mathbb{Z})$. Show that there exists a continuous map $f \colon S^1 \longrightarrow X$ with $\alpha \in \operatorname{im} H_1(f;\mathbb{Z})$.
2. How can this be reformulated in terms of the Hurewicz homomorphism?

Bonus Problem (spectral sequences/Spektralsequenzen)**.** Look up the corresponding terms in the literature and answer the following questions:

1. What is a homological spectral sequence?
2. What is (bounded) convergence of a homological spectral sequence?
3. Formulate a spectral sequence and a convergence theorem for (bounded) filtrations of chain complexes.
4. How can this spectral sequence be used to obtain the long exact homology sequence for short exact sequences of chain complexes as a special case?

Illustrate your answers with schematic sketches!

Please turn over

The following exercises will increase your algebraic topology XP!

Bonus Problem (π_0)**.** Give an explicit geometric description of the functor $\pi_0\colon \mathsf{Top}_* \longrightarrow \mathsf{Set}$. Illustrate your arguments with suitable drawings.

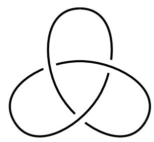
Bonus Problem (trivial elements in homotopy groups)**.** Let $n \in \mathbb{N}$, let (X, x_0) be a pointed space and let $f \in \operatorname{map}_*((S^n, e_1^n), (X, x_0))$. Show that $[f]$ is trivial in $\pi_n(X, x_0)$ if and only if $f\colon S^n \longrightarrow X$ can be extended to a continuous map $D^{n+1} \longrightarrow X$. Illustrate!

Bonus Problem ($\mathbb{R}P^\infty$)**.** We consider $S^\infty := \bigcup_{n\in\mathbb{N}} S^n$, where each sphere is included in the next one as equator; we endow S^∞ with the corresponding colimit topology. The *infinite real projective space* $\mathbb{R}P^\infty$ is then defined as the quotient of S^∞ by the corresponding antipodal action of $\mathbb{Z}/2$.

1. Calculate the fundamental group of $\mathbb{R}P^\infty$.
2. What can you say about the higher homotopy groups of $\mathbb{R}P^\infty$?

Bonus Problem (knots/Knoten)**.** A *knot* is a continuous injective map of type $S^1 \longrightarrow \mathbb{R}^3$. The *knot complement* of a knot $K\colon S^1 \longrightarrow \mathbb{R}^3$ is the complement $\mathbb{R}^3 \setminus K(S^1)$.

1. Does it make sense to study knots by considering knots up to homotopy of maps? Look up a suitable notion of equivalence of knots in the literature!
2. Does it make sense to study knots by considering ordinary homology of knot complements?



Bonus Problem (transfer for finite coverings/Transfer)**.** Let $p\colon X \longrightarrow Y$ be a d-sheeted covering of topological spaces with $d \in \mathbb{N}$. For $k \in \mathbb{N}$ and a singular simplex $\sigma \in \operatorname{map}(\Delta^k, X)$ we define

$$p^{-1}(\sigma) := \bigl\{\tau \in \operatorname{map}(\Delta^k, Y) \bigm| p \circ \tau = \sigma\bigr\}.$$

We then define the *transfer map*

$$\begin{aligned} T_k\colon C_k(X) &\longrightarrow C_k(Y) \\ \operatorname{map}(\Delta^k, X) \ni \sigma &\longmapsto \sum_{\tau \in p^{-1}(\sigma)} \tau \end{aligned}$$

for all $k \in \mathbb{N}$; furthermore, we let $T_k := 0$ for all $k \in \mathbb{Z}_{<0}$.

1. Prove that $(T_k)_{k\in\mathbb{Z}}$ is a well-defined chain map $C(X) \longrightarrow C(Y)$.
2. For a ring R with unit, determine $H_*(p; R) \circ H_*(R \otimes_{\mathbb{Z}} T)$.
3. What does this imply for $H_*(\mathbb{R}P^n; \mathbb{Z})$ and $H_*(\mathbb{R}P^n; \mathbb{R})$ if $n \in \mathbb{N}_{>0}$?

Submission before January 8, 2016, 10:00, in the mailbox

Merry Christmas and a Happy New Year!