# Algebraic Topology I – Exercises

Prof. Dr. C. Löh Sheet 11, January 8

**Exercise 1** (chain homotopy equivalences)**.** Let $R$ be a ring with unit, let $Z \in \operatorname{Ob}({}_R\mathsf{Mod})$, and let $C, D \in \operatorname{Ob}({}_{\mathbb{Z}}\mathsf{Ch})$. Prove or disprove:

1. If $f\colon C \longrightarrow D$ is a chain homotopy equivalence in ${}_{\mathbb{Z}}\mathsf{Ch}$, then the tensor product $Z \otimes_{\mathbb{Z}} f\colon Z \otimes_{\mathbb{Z}} C \longrightarrow Z \otimes_{\mathbb{Z}} D$ is a chain homotopy equivalence in ${}_R\mathsf{Ch}$.
2. If $Z \otimes_{\mathbb{Z}} f\colon Z \otimes_{\mathbb{Z}} C \longrightarrow Z \otimes_{\mathbb{Z}} D$ is a chain homotopy equivalence in ${}_R\mathsf{Ch}$, then $f\colon C \longrightarrow D$ is a chain homotopy equivalence in ${}_{\mathbb{Z}}\mathsf{Ch}$.

**Exercise 2** (the standard resolution/die Standard-Auflösung)**.** Let $G$ be a group. We consider the chain complex $C \in \operatorname{Ob}({}_{\mathbb{Z}}\mathsf{Ch})$ with the chain modules given by

$$C_n := \begin{cases} \bigoplus_{g \in G^{n+1}} \mathbb{Z} \cdot g & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$$

for all $n \in \mathbb{Z}$ and the boundary operator given by

$$\begin{aligned} \partial_n\colon C_n &\longrightarrow C_{n-1} \\ G^{n+1} \ni g &\longmapsto \sum_{j=0}^{n} (-1)^j \cdot (g_0, \dots, g_{j-1}, g_{j+1}, \dots, g_n) \end{aligned}$$

for all $n \in \mathbb{N}_{>0}$. Show that $C$ is chain homotopy equivalent to the chain complex $D$ concentrated in degree 0 with $D_0 \cong_{\mathbb{Z}} \mathbb{Z}$.

**Exercise 3** (the $\ell^1$-semi-norm on singular homology)**.** Let $X$ be a topological space and let $k \in \mathbb{N}$. Let $|\cdot|_1$ be the $\ell^1$-norm on $C_k(X;\mathbb{R})$ with respect to the $\mathbb{R}$-basis of $C_k(X;\mathbb{R})$ that consists of all singular $k$-simplices of $X$. We then define $\|\cdot\|_1\colon H_k(X;\mathbb{R}) \longrightarrow \mathbb{R}_{\geq 0}$ by

$$\|\alpha\|_1 := \inf\bigl\{|c|_1 \bigm| c \in C_k(X;\mathbb{R}),\ \partial_k c = 0,\ [c] = \alpha \in H_k(X;\mathbb{R})\bigr\}$$

for all $\alpha \in H_k(X;\mathbb{R})$.

1. Show that $\|\cdot\|_1$ is a semi-norm on $H_k(X;\mathbb{R})$.
2. Let $f\colon X \longrightarrow Y$ be a homotopy equivalence. Show that the induced homomorphism $H_k(f;\mathbb{R})\colon H_k(X;\mathbb{R}) \longrightarrow H_k(Y;\mathbb{R})$ is isometric with respect to $\|\cdot\|_1$.

*Please turn over*

**Exercise 4** (singular homology of ascending unions)**.** Let $R$ be a ring with unit and let $Z \in \mathrm{Ob}({}_R\mathsf{Mod})$. Let $X$ be a topological space and let $(X_n)_{n\in\mathbb{N}}$ be an ascending sequence of subspaces of $X$ with $\bigcup_{n\in\mathbb{N}} X_n^\circ = X$. Use a compactness argument to prove the following: For all $k \in \mathbb{Z}$ the inclusion maps $(X_n \hookrightarrow X)_{n\in\mathbb{N}}$ induce an isomorphism

$$\operatorname*{colim}_{n\in\mathbb{N}} H_k(X_n; Z) \longrightarrow H_k(X; Z)$$

of left $R$-modules.

*Hints.* We write

$$\operatorname*{colim}_{n\in\mathbb{N}} H_k(X_n; Z) := \Bigl(\bigoplus_{n\in\mathbb{N}} H_k(X_n; Z)\Bigr) \Big/ \sim,$$

where "$\sim$" is the equivalence relation that is generated by

$$\forall_{n\in\mathbb{N}} \quad \forall_{m\in\mathbb{N}} \quad \forall_{\alpha\in H_k(X_n;Z)} \quad \alpha \sim H_k(i_{n,m}; Z)(\alpha) \in H_k(X_m; Z)$$

and where $i_{n,m} \colon X_n \longrightarrow X_m$ are the corresponding inclusions.

**Bonus Problem** (singular homology of weakly contractible spaces)**.** A topological space $X$ is *weakly contracible* (schwach kontraktibel) if there is an $x_0 \in X$ such that for all $n \in \mathbb{N}$ the set $\pi_n(X, x_0)$ has exactly one element.

1. Let $X$ be weakly contractible. Show that the constant map $X \longrightarrow \bullet$ induces an isomorphism $H_*(X; \mathbb{Z}) \longrightarrow H_*(\bullet; \mathbb{Z})$.

   *Hints.* Inductively, replace the singular chain complex $C(X)$ by subcomplexes that are generated by singular simplices that are constant on low-dimensional faces.

2. What does this imply for the singular homology with $\mathbb{Z}$-coefficients of the Warsaw circle?