Algebraic Topology I – Exercises

Prof. Dr. C. Löh — Sheet 12, January 15

Exercise 1 (the cone operator)**.** Let $k \in \mathbb{N}$ and let $\sigma\colon \Delta^k \longrightarrow \bigoplus_{\mathbb{N}} \mathbb{R}$ be a singular simplex. Prove or disprove:

1. For all $v, w \in \bigoplus_{\mathbb{N}} \mathbb{R}$ we have $v * (w * \sigma) = w * (v * \sigma)$.
2. For all $v \in \bigoplus_{\mathbb{N}} \mathbb{R}$ we have $B_{k+1}(v * \sigma) = v * B_k(\sigma)$.

Exercise 2 (diameter of subdivisions of affine simplices)**.** Let $k \in \mathbb{N}$. A singular simplex $\sigma\colon \Delta^k \longrightarrow \bigoplus_{\mathbb{N}} \mathbb{R}$ is *affine* if

$$\sigma(t_0, \dots, t_k) = \sum_{j=0}^{k} t_j \cdot \sigma(e_{j+1})$$

holds for all $(t_0, \dots, t_k) \in \Delta^k \subset \mathbb{R}^{k+1}$, where $e_1, \dots, e_{k+1} \in \mathbb{R}^{k+1}$ are the standard unit vectors. For $A \subset \bigoplus_{\mathbb{N}} \mathbb{R}$ we call

$$\operatorname{diam} A := \sup_{x,y \in A} \|x - y\|_2 \in \mathbb{R}_{\geq 0} \cup \{\infty\}$$

the *diameter of A*.

1. Prove that for all affine simplices $\sigma\colon \Delta^k \longrightarrow \bigoplus_{\mathbb{N}} \mathbb{R}$ we have
$$\operatorname{diam}\bigl(\sigma(\Delta^k)\bigr) = \max_{j,j' \in \{0,\dots,k\}} \bigl\|\sigma(e_{j+1}) - \sigma(e_{j'+1})\bigr\|_2.$$
2. Prove that if $\sigma\colon \Delta^k \longrightarrow \bigoplus_{\mathbb{N}} \mathbb{R}$ is affine, then all simplices τ occuring in the barycentric subdivision $B_k(\sigma)$ satisfy
$$\operatorname{diam}\bigl(\tau(\Delta^k)\bigr) \leq \frac{k}{k+1} \cdot \operatorname{diam}\bigl(\sigma(\Delta^k)\bigr).$$

Exercise 3 (barycentric subdivision and homotopy groups?)**.** Let (X, x_0) be a pointed space and let $A, B \subset X$ with $\overline{B} \subset A^\circ$ and $x_0 \in A \setminus B$. Recall that for all $k \in \mathbb{N}$ there is a homeomorphism $S^k \cong \Delta^k/\partial\Delta^k$; hence, we can view elements of $\pi_k(X, x_0)$ as represented by continuous maps $\Delta^k \longrightarrow X$.

1. Look up the definition of relative homotopy groups in the literature.
2. Give reasons why the barycentric subdivision technique does *not* also provide a proof that the inclusion $(X \setminus B, A \setminus B) \hookrightarrow (X, A)$ induces for all $k \in \mathbb{N}$ a bijection $\pi_k(X \setminus B, A \setminus B, x_0) \longrightarrow \pi_k(X, A, x_0)$.

Please turn over

Exercise 4 (Slitherlink)**.** A *Slitherlink puzzle* consists of a square grid; some of the squares carry numbers. The goal is to produce a closed loop out of the edges of the grid that is compatible with the given numbers in the following sense:

SL 1 Neighbouring grid points are joined by vertical or horizontal edges in such a way that we obtain a closed loop.

SL 2 The numbers indicate how many of the edges of a given square belong to the loop. For empty squares the number of edges in the loop is not specified.

SL 3 The loop does not have any self-intersections or branches.

1. Solve the following Slitherlink puzzle:

2							3				1
	3		3	3	3	3			1		
		1					1	3	3	3	3
			1		1			1			
2		3			3					3	
1		3			3	2					2

2. Which theorem from topology can give global strategies for solving Slitherlink puzzles? Give an example that illustrates this strategy.

Bonus Problem (Alexander horned sphere/gehörnte Alexander-Sphäre)**.**

1. Carry out the construction of the Alexander horned sphere $S \subset \mathbb{R}^3$ in full detail.

2. Assume that there exist an infinitely decreasing hierarchy of Duplo, Lego, ... bricks. Explain how to build the Alexander horned sphere using such bricks.

3. Prove that the complement $\mathbb{R}^3 \setminus S$ is *not* simply connected.

Submission before January 22, 10:00, in the mailbox