Algebraic Topology I – Exercises

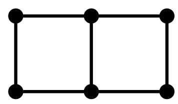
Prof. Dr. C. Löh   Sheet 14, January 29

Exercise 1 (open cells)**.** Let X be a CW-complex and let $e \subset X$ be an open cell. Prove or disprove:

1. Then e is an open subset of X.
2. If $f \colon X \longrightarrow X$ is a cellular map, then there is an open cell e' of X with $f(e) \subset e'$.

Exercise 2 (spanning trees)**.** Let X be a one-dimensional CW-complex, i.e., $X_n = X_1$ for all $n \in \mathbb{N}_{\geq 2}$. A *spanning tree of* X is a contractible subcomplex T of X with $X_0 \subset T$.

1. Determine two non-homeomorphic spanning trees of the following one-dimensional CW-complex:



2. Prove that every non-empty path-connected one-dimensional CW-complex admits a spanning tree. Zorn's lemma might help!

Hints. A *subcomplex* of a CW-complex X is a subspace $A \subset X$ with the following property: for all open cells $e \subset X$ of X with $e \cap A \neq \emptyset$ we have $\overline{e} \subset A$. In this case, $(A \cap X_n)_{n \in \mathbb{N}}$ is a CW-structure on A.

Exercise 3 (products of CW-complexes)**.** Solve two out of the following four problems:

1. Let X be a CW-complex and let Y be a finite CW-complex. Prove that
$$\Bigl(\bigcup_{k \in \{0,\dots,n\}} X_k \times Y_{n-k}\Bigr)_{n \in \mathbb{N}}$$
is a CW-structure on $X \times Y$.

   Hints. You may use the following facts: All finite CW-complexes are compact. If $\pi \colon Z \longrightarrow Z'$ is a quotient map (i.e., π is surjective and Z' carries the quotient topology induced by π) and K is a compact space, then also $\pi \times \mathrm{id}_K \colon Z \times K \longrightarrow Z' \times K$ is a quotient map.

2. Construct CW-structures on $\mathbb{R}P^2 \times S^3$ and $S^2 \times \mathbb{R}P^3$.
3. Let h be singular homology with $\mathbb{Z}$-coefficients. Prove that
$$H_5^h(\mathbb{R}P^2 \times S^3) \not\cong_{\mathbb{Z}} H_5^h(S^2 \times \mathbb{R}P^3).$$
4. Show that $\mathbb{R}P^2 \times S^3$ and $S^2 \times \mathbb{R}P^3$ have isomorphic homotopy groups.

Exercise 4 (coverings of CW-complexes)**.** Let X be a CW-complex with CW-structure $(X_n)_{n \in \mathbb{N}}$ and let $p \colon Y \longrightarrow X$ be a covering map. Prove that the subspaces $\bigl(p^{-1}(X_n)\bigr)_{n \in \mathbb{N}}$ form a CW-structure on the covering space Y.

Please turn over

Bonus Problem (fundamental groups of CW-complexes)**.**

1. Prove that all non-empty path-connected one-dimensional CW-complexes have free fundamental group.

   Hints. Contract a spanning tree (see Exercise 2) to a point ...

2. How can one determine the rank of the fundamental group of a path-connected finite one-dimensional CW-complex?

3. How can one describe the fundamental group of a path-connected two-dimensional CW-complex in terms of attaching maps?

4. Prove that the fundamental group of a path-connected CW-complex depends only on the 2-skeleton.

---

no submission