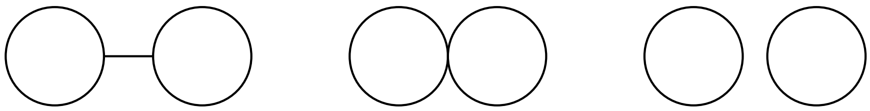
# Algebraic Topology I – Exercises

Prof. Dr. C. Löh Sheet 2, October 23, 2015

**Exercise 1** (functorial simplices?)**.** Prove or disprove:

1. There exists a functor $\Delta_{\mathsf{Top}} \colon \Delta \longrightarrow \mathsf{Top}$ that satisfies $\Delta_{\mathsf{Top}}(\Delta(n)) = \Delta^n$ for all $n \in \mathbb{N}$.
2. There exists a functor $\Delta_{\mathsf{Ab}} \colon \Delta \longrightarrow \mathsf{Ab}$ that satisfies $\Delta_{\mathsf{Ab}}(\Delta(n)) = \mathbb{Z}^{n+1}$ for all $n \in \mathbb{N}$.

**Exercise 2** (homotopy equivalences)**.** We consider the following three subspaces of $\mathbb{R}^2$. State and prove the classification of these spaces up to homotopy equivalence.



*Hints.* Prove that path-connectedness is preserved under homotopy equivalences.

**Exercise 3** (exponential law and homotopy)**.** Let $X$ be a locally compact topological space and let $Y$ be a topological space.

1. Prove that the map
$$\begin{aligned} \operatorname{map}\bigl(X \times [0,1], Y\bigr) &\longrightarrow \operatorname{map}\bigl([0,1], \operatorname{map}(X,Y)\bigr) \\ h &\longmapsto \bigl(t \mapsto h(\,\cdot\,,t)\bigr) \end{aligned}$$
is well-defined and bijective; here, $\operatorname{map}(X,Y)$ carries the compact open topology.
2. Use the above exponential law to reinterpret the notion of homotopy between maps.

*Hints.* A topological space $X$ is *locally compact* if the following holds: for every $x \in X$ and every open neighbourhood $U$ of $x$ there is a compact neighbourhood $K \subset X$ of $x$ with $K \subset U$. (Please note that in the literature sometimes also weaker conditions are associated with locally compactness.)

If $X$ and $Y$ are topological spaces, then the *compact open topology* (kompakt-offene Topologie) on $\operatorname{map}(X,Y)$ is the topology that is generated by all sets of the shape
$$\bigl\{f \in \operatorname{map}(X,Y) \bigm| f(K) \subset U\bigr\},$$
where $K \subset X$ is compact and $U \subset Y$ is open.

As a first step, prove that if $Z$ and $Y$ are topological spaces where $Z$ is locally compact, then the evaluation map
$$\begin{aligned} \operatorname{map}(Z,Y) \times Z &\longrightarrow Y \\ (f,z) &\longmapsto f(z) \end{aligned}$$
is continuous.

*Please turn over*

**Exercise 4** (Yoneda lemma)**.**

1. Look up the *Yoneda lemma* in the literature and formulate it; please do not forget to cite your sources properly.

2. Prove the Yoneda lemma.

**Bonus Problem** (Galois theory)**.** Formulate the fundamental theorem of Galois theory as isomorphism between suitable categories and derive your version from the classical formulation of the fundamental theorem of Galois theory.

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Submission before October 30, 10:00, in the mailbox