Algebraic Topology I – Exercises

Prof. Dr. C. Löh Sheet 3, October 30, 2015

Exercise 1 (homotopy invariant functors)**.** Let $F\colon \mathsf{Top} \longrightarrow \mathsf{Ab}$ be a homotopy invariant functor. Prove or disprove:

1. Then $F(\mathbb{R}P^{2015}) \cong 0$.
2. If $F(D^{2016}) \cong \mathbb{Z}$, then $F(S^{2015}) \not\cong \mathbb{Q}$.

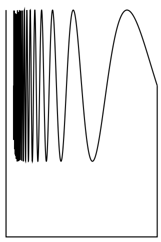
Exercise 2 (invariance of dimension)**.** In this exercise, you may assume that the theorem on existence of "interesting" homotopy invariant functors holds. Prove *invariance of domain*, i.e., show that for all $n, m \in \mathbb{N}$ we have $\mathbb{R}^n \cong \mathbb{R}^m$ if and only if $n = m$.
Hints. Try to get spheres involved!

Exercise 3 (invariance of the boundary)**.** In this exercise, you may assume that the theorem on existence of "interesting" homotopy invariant functors holds.

1. Let $n \in \mathbb{N}_{>0}$ and let $H^n := \{x \in \mathbb{R}^n \mid x_n \geq 0\}$ be the upper half-space (with respect to the subspace topology of $\mathbb{R}^n$). Prove *invariance of the boundary*, i.e., show that there is no open neighbourhood of 0 in H^n that is homeomorphic to the open unit ball $(D^n)^\circ$ in $\mathbb{R}^n$.

   Hints. As first step, prove the following: If $U \subset H^n$ is an open neighbourhood of 0, then $U \setminus \{0\}$ is homotopy equivalent to U.
2. Conclude that the Möbius strip and the boring strip $S^1 \times [0, 1]$ are *not* homeomorphic.

Exercise 4 (Warsaw circle/Warschauer Kreis)**.** The topological space



$$\begin{aligned} W := &\bigl\{(x, \sin(2 \cdot \pi/x)) \bigm| x \in (0, 1]\bigr\} \\ &\cup \bigl(\{1\} \times [-2, 0]\bigr) \cup \bigl([0, 1] \times \{-2\}\bigr) \cup \bigl(\{0\} \times [-2, 1]\bigr) \end{aligned}$$

(endowed with the subspace topology of $\mathbb{R}^2$) is called *Warsaw circle*. Prove that for every baspoint $w_0 \in W$ the fundamental group $\pi_1(W, w_0)$ is trivial.
Hints. Show first that no loop in W can cross the "gap."

Bonus Problem (Nash equilibria/Nash-Gleichgewichte)**.** Look up the notion of Nash equilibria in Game Theory. Prove the existence of Nash equilibria using the Brouwer fixed point theorem.

Submission before November 6, 10:00, in the mailbox