Algebraic Topology I – Exercises

Prof. Dr. C. Löh — Sheet 4, November 6, 2015

Exercise 1 (spheres and π_n)**.** Let $n \in \mathbb{N}$. Prove or disprove:

1. If $\pi_n(S^n, e_1^n)$ consists of a single element, then for every pointed topological space (X, x_0) also $\pi_n(X, x_0)$ consists of a single element.
2. If $\pi_n(S^n, e_1^n)$ has at least two elements, then for every pointed space (X, x_0) also $\pi_n(X, x_0)$ has at least two elements.

Exercise 2 (Lebesgue Lemma)**.** Let (X, d) be a compact metric sapce and let $(U_i)_{i \in I}$ be an open cover of X. Prove that there exists an $\varepsilon \in \mathbb{R}_{>0}$ with the following property: for every $x \in X$ there is an $i \in I$ such that the open ε-ball $U(\varepsilon, x)$ in X around x is contained in U_i.

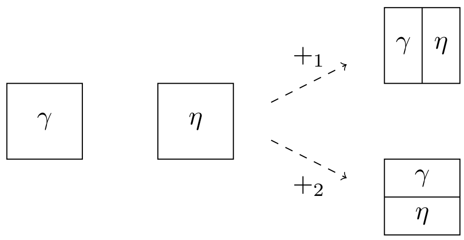
Exercise 3 (π_1 and contractibility)**.** Let X be a topological space.

1. Let $\gamma \colon S^1 \longrightarrow X$ be a null-homotopic map and let $x_0 := \gamma(1)$. Show that then $[\gamma]_*$ is trivial in $\pi_1(X, x_0)$. Illustrate your argument in a suitable way!
2. Conclude: If X is contractible (but not necessarily pointed contractible!) and $x_0 \in X$, then $\pi_1(X, x_0)$ is the trivial group.

Exercise 4 (a group structure on π_n)**.** Let $n \in \mathbb{N}_{>0}$ and let $\square^n := [0,1]^n$ be the unit n-cube. Similarly to Exercise 3 on Sheet 0 one can prove that there is a homeomorphism $\varphi_n \colon \square^n / \partial\square^n \longrightarrow S^n$ with $\varphi_n(\partial\square^n) = e_1^n$; in the following, we will parametrise points in S^n by $\square^n$ using this homeomorphism φ_n. For pointed spaces (X, x_0) and $j \in \{1, \dots, n\}$ we define the composition

$$
\begin{aligned}
+_j \colon \pi_n(X, x_0) \times \pi_n(X, x_0) &\longrightarrow \pi_n(X, x_0) \\
\bigl([\gamma]_*, [\eta]_*\bigr) &\longmapsto \left[\square^n \ni t \mapsto \begin{cases} \gamma(t_1, \dots, 2 \cdot t_j, \dots, t_n) & \text{if } t_j \in [0, 1/2] \\ \eta(t_1, \dots, 2 \cdot t_j - 1, \dots, t_n) & \text{if } t_j \in [1/2, 1] \end{cases}\right]_*
\end{aligned}
$$

on $\pi_n(X, x_0) = \bigl[(S^n, e_1^n), (X, x_0)\bigr]_*$.



Similar arguments as in the case of $\pi_1(X, x_0)$ show that $+_j$ indeed yields a well-defined functorial group structure on $\pi_n(X, x_0)$.

1. Prove that $+_j = +_1$ for all $j \in \{1, \dots, n\}$.
2. Let $n \geq 2$. Prove that $\pi_n(X, x_0)$ is an Abelian group with respect to $+_1$.

Please turn over

Bonus Problem (fundamental theorem of algebra)**.** Prove that every non-constant polynomial in $\mathbb{C}[X]$ has at least one root in $\mathbb{C}$, using the fact that

$$\begin{aligned} \mathbb{Z} &\longrightarrow \pi_1(S^1, 1) \\ n &\longmapsto [z \mapsto z^n]_* \end{aligned}$$

is a group isomorphism (where we view S^1 as the set of complex numbers of norm 1).

Hints. Show that a non-constant polynomial $p \in \mathbb{C}[X]$ without roots in $\mathbb{C}$ would yield a map $S^1 \longrightarrow S^1$ that is both null-homotopic and homotopic to the map $S^1 \longrightarrow S^1$, $z \mapsto z^{\deg p}$, ...
