Algebraic Topology I – Exercises

Prof. Dr. C. Löh — Sheet 5, November 13, 2015

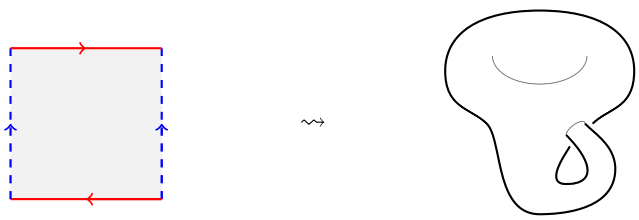
Hints. In the following, you may use that

$$\begin{aligned} \mathbb{Z} &\longrightarrow \pi_1(S^1, 1) \\ n &\longmapsto \bigl[\mathbb{C} \supset S^1 \ni z \mapsto z^n \in S^1 \subset \mathbb{C}\bigr]_* \end{aligned}$$

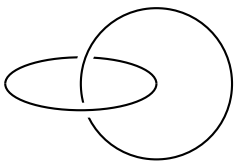
is a group isomorphism.

Exercise 1 (coverings by tori?)**.** Prove or disprove:

1. There is a covering $S^1 \times S^1 \longrightarrow S^1$.
2. There is a covering $S^1 \times S^1 \longrightarrow K$, where $K := [0,1] \times [0,1]/\sim$ is the *Klein Bottle* (Kleinsche Flasche), which is defined as the following quotient space:



Exercise 2 (Houdini?)**.** A rubber ring and a steel ring are entangled in $\mathbb{R}^3$ as illustrated:



Can these two rings be unlinked in $\mathbb{R}^3$ by deforming the rubber ring (without cutting it)?

1. Model the situation above, using appropriate topological terms.
2. Answer the question above (with a full proof) with respect to your model.
3. What changes if we look at the same rings in $\mathbb{R}^4$ instead of $\mathbb{R}^3$?

Illustrate your arguments in a suitable way!

Exercise 3 (fundamental group of the real projective plane)**.** Let $x_0 \in \mathbb{R}P^2$. Use the Theorem of Seifert and van Kampen to prove that

$$\pi_1(\mathbb{R}P^2, x_0) \cong \mathbb{Z}/2$$

and give a geometric description of a loop that represents the generator.

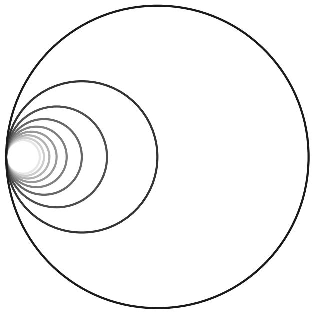
Please turn over

Exercise 4 (pretzel coverings)**.** Let $(B, b) := (S^1, 1) \vee (S^1, 1)$.

1. Draw two connected double coverings (i.e., with two sheets) of (B, b) that are *not* isomorphic in $\mathrm{Cov}_{(B,b)}$ (and sketch a proof of this fact). Moreover, describe the maps on π_1 induced by these coverings geometrically.

2. Draw a connected triple covering (i.e., with three sheets) of (B, b) whose deck transformation group does *not* act transitively on the fibres (and sketch a proof of this fact).

Bonus Problem (Hawaiian earring)**.** We consider the following subspace H of $\mathbb{R}^2$ with the subspace topology, the so-called *Hawaiian earring*:



$$H := \bigcup_{n \in \mathbb{N}_{>0}} \{x \in \mathbb{R}^2 \mid d\big(x, (1/n, 0)\big) = 1/n\}$$

Prove that $\pi_1(H, 0)$ is uncountable and illustrate your arguments in a suitable way. Is $(H, 0)$ (pointed) homotopy equivalent to $\bigvee_{\mathbb{N}}(S^1, 1)$?

Submission before November 20, 10:00, in the mailbox