Algebraic Topology I – Exercises

Prof. Dr. C. Löh — Sheet 6, November 20, 2015

Hints. In the following, you may use that

$$\begin{aligned} \mathbb{Z} &\longrightarrow \pi_1(S^1, 1) \\ n &\longmapsto \bigl[\mathbb{C} \supset S^1 \ni z \mapsto z^n \in S^1 \subset \mathbb{C}\bigr]_* \end{aligned}$$

is a group isomorphism.

Exercise 1 (universal coverings?)**.** Prove or disprove:

1. The Hawaiian earring (see the Bonus Exercise on Sheet 5 for a definition) admits a universal covering.
2. The Warsaw circle (see Exercise 4 on Sheet 3 for a definition) admits a non-trivial covering.

   Hints. You may use that $\bigl\{(x, \sin(2 \cdot \pi/x)) \bigm| x \in (0,1]\bigr\} \cup \bigl(\{0\} \times [-1,1]\bigr)$ (with the subspace topology of $\mathbb{R}^2$) is connected.

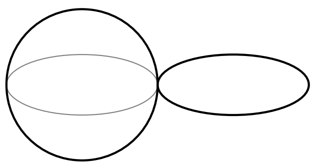
Exercise 2 (action of the fundamental group on the fibre)**.** Let $p\colon Y \longrightarrow X$ be a covering and let $x_0 \in X$.

1. Prove that

   $$\begin{aligned} p^{-1}(x_0) \times \pi_1(X, x_0) &\longrightarrow p^{-1}(x_0) \\ \bigl(y, [\gamma]_*\bigr) &\longmapsto \widetilde{\gamma}(1) \\ &\qquad \text{where } \widetilde{\gamma}\colon [0,1] \to Y \text{ is the } p\text{-Lift} \\ &\qquad \text{of } [0,1] \to X,\ t \mapsto \gamma([t]) \text{ with } \widetilde{\gamma}(0) = y \end{aligned}$$

   is a well-defined right action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$.
2. Let $y \in p^{-1}(x_0)$. Show that $\pi_1(p)(\pi_1(Y, y))$ is the stabiliser of y with respect to this action.
3. Show that this action is transitive if and only if the fibre $p^{-1}(x_0)$ is contained in a single path-connected component of Y.

Exercise 3 (large homotopy groups)**.** Let $(X, x_0) := (S^2, e_1^2) \vee (S^1, 1)$.



1. Give a simple geometric description of the universal covering of X and prove that this indeed is a universal covering of X.
2. Let $k \in \mathbb{N}_{\geq 2}$ with the property that $\pi_k(S^2, e_1^2)$ is non-trivial. Prove that then $\pi_k(X, x_0)$ is *not* finitely generated (as Abelian group).

Please turn over

Exercise 4 (existence of universal coverings)**.** Let X be a path-connected, locally path-connected, semi-locally simply connected topological space and let $x_0 \in X$. We define

$$\widetilde{X} := \operatorname{map}_*\bigl(([0,1],0),(X,x_0)\bigr) / \sim,$$

where two pointed paths $\gamma, \eta \in \operatorname{map}_*\bigl(([0,1],0),(X,x_0)\bigr)$ satisfy $\gamma \sim \eta$ if and only if the closed path $\gamma * \overline{\eta}$ represents the trivial element of $\pi_1(X,x_0)$. We equip $\widetilde{X}$ with the quotient topology of the subspace topology of the compact open topology on $\operatorname{map}([0,1],X)$ (Exercise 3 on Sheet 2).

1. Show that $\widetilde{X}$ is path-connected.
2. Show that

$$\begin{aligned} p \colon \widetilde{X} &\longrightarrow X \\ [\gamma]_\sim &\longmapsto \gamma(1) \end{aligned}$$

   is a covering map.

   Hints. You can show this by hand or construct a suitable properly discontinuous action of $\pi_1(X,x_0)$ on $\widetilde{X}$ ...

3. Show that $\widetilde{X}$ is simply connected.

   Hints. You can show this by hand or use a suitable argument from covering theory.

Bonus Problem (isometry groups; requires some background in Riemannian geometry)**.** Let (M,g) be a compact connected Riemannian manifold and let $p \colon \widetilde{M} \longrightarrow M$ be its universal covering (with the smooth structure induced by M). Let $\widetilde{g} := p^* g$ be the induced Riemannian metric on $\widetilde{M}$. Moreover, let $x_0 \in M$.

1. Prove that the deck transformation action of $\pi_1(X,x_0)$ (with respect to some chosen point in $p^{-1}(x_0)$) on $(\widetilde{M},\widetilde{g})$ is isometric.
2. Prove that this defines an embedding of $\pi_1(X,x_0)$ into the isometry group of $(\widetilde{M},\widetilde{g})$ as a uniform lattice.

   Hints. Here, we equip the isometry group with the compact open topology. A subgroup of a topological group is a *uniform lattice* (uniformes Gitter) if this subgroup is discrete (with respect to the subspace topology) and the quotient space (with respect to the quotient topology) is compact.

Submission before November 27, 10:00, in the mailbox