Algebraic Topology I – Exercises

Prof. Dr. C. Löh — Sheet 7, November 27, 2015

Exercise 1 (2 out of 3 ?)**.** Let X, Y, Z be non-empty path-connected and locally path-connected spaces, let $p\colon Y \longrightarrow X$ be a covering, and let $q\colon Z \longrightarrow X$, $r\colon Y \longrightarrow Z$ be continuous maps with $q \circ r = p$. Prove or disprove:

1. If r is a covering map, then also q is a covering map.
2. If q is a covering map, then also r is a covering map.

Exercise 2 (The Borsuk-Ulam theorem in dimension 2)**.** A map $f\colon S^2 \longrightarrow S^1$ is *antipodal* if $f(-x) = -f(x)$ holds for all $x \in S^2$.

1. Show that there is no continuous antipodal map $S^2 \longrightarrow S^1$.
   Hints. Try to use $\mathbb{R}P^2$ and consider a path in S^2 from e_1^2 to $-e_1^2$.
2. Prove the *Borsuk-Ulam theorem in dimension* 2: If $f\colon S^2 \longrightarrow \mathbb{R}^2$ is continuous, then there exists $x \in S^2$ with $f(x) = f(-x)$.
3. Conclude that there is no subspace in $\mathbb{R}^2$ that is homeomorphic to S^2.
4. Give a real-world interpretation of the Borsuk-Ulam theorem by viewing S^2 as a model of a ball or a planet.

Exercise 3 (the Heisenberg manifold)**.** We consider the *Heisenberg group*

$$H := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{Z} \right\} \subset H_{\mathbb{R}} := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\} \subset \mathrm{SL}_3(\mathbb{R})$$

and the *Heisenberg manifold* $M := H \setminus H_{\mathbb{R}}$, where H acts on $H_{\mathbb{R}}$ by matrix multiplication. Give detailed proofs for your answers to the following questions:

1. What is the fundamental group of M ?
2. Is there a covering of the 3-torus by M ?
3. Is there a covering of M by the 3-torus ?
4. Does M admit a path-connected regular covering whose deck transformation group is isomorphic to $\mathbb{Z}^2$?

Exercise 4 (residually finite groups/residuell endliche Gruppen)**.** A group G is *residually finite* if the following holds: For every $g \in G \setminus \{e\}$ there exists a finite group F and a group homomorphism $\varphi\colon G \longrightarrow F$ with $\varphi(g) \neq e$.

1. Let (X, x_0) be a pointed space that admits a universal covering. Give an equivalent characterisation of residual finiteness of $\pi_1(X, x_0)$ in terms of coverings of (X, x_0) and paths.
2. Illustrate these concepts for the circle $(S^1, 1)$.

Please turn over

Bonus Problem (classifying spaces of torsion-free groups)**.** Let G be a group and let $F(G)$ denote the set of all finite subsets of G. For $S \in F(G)$ we write

$$\Delta_G(S) := \Bigl\{ f\colon G \longrightarrow [0,1] \Bigm| f|_{G\setminus S} = 0 \text{ and } \sum_{s\in S} f(s) = 1 \Bigr\}.$$

We write $\Delta(G) := \bigcup_{S\in F(G)} \Delta_G(S)$ with the corresponding colimit topology (i.e., a subset of $\Delta(G)$ is closed if and only if the intersection with $\Delta_G(S)$ is closed for all $S \in F(G)$, where $\Delta_G(S)$ carries the Euclidean topology). Furthermore, we consider the continuous left action

$$\begin{aligned} G \times \Delta(G) &\longrightarrow \Delta(G) \\ (g, f) &\longmapsto \bigl(h \mapsto f(g^{-1}\cdot h)\bigr) \end{aligned}$$

of G on $\Delta(G)$.

1. Show that $\Delta(G)$ is pointed contractible with respect to every basepoint.
2. Determine the fundamental group of $G\setminus\Delta(G)$ in the case that the group G is torsion-free.
3. Extend this construction to an interesting functor $\mathsf{Group} \longrightarrow \mathsf{Top}$.

Submission before December 4, 10:00, in the mailbox