Algebraic Topology I – Exercises

Prof. Dr. C. Löh | Sheet 8, December 4, 2015

Hints. In the following, we write $\bullet := \{\emptyset\}$ for the canonical one-point space. Moreover, let R be a ring with unit and let $\big((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\big)$ be a homology theory on Top^2 with values in ${}_R\mathsf{Mod}$.

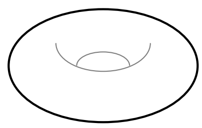
Exercise 1 (suspensions)**.** Let $X \neq \emptyset$ be a path-connected space. Prove or disprove:

1. Then ΣX is simply connected.
2. If $p\colon Y \longrightarrow X$ is a covering map, then $\Sigma p\colon \Sigma Y \longrightarrow \Sigma X$ is a covering map.

Exercise 2 (more homology theories)**.**

1. Let $d \in \mathbb{Z}$. Show that $\big((h_{k+d})_{k\in\mathbb{Z}}, (\partial_{k+d})_{k\in\mathbb{Z}}\big)$ is a homology theory on Top^2 with values in ${}_R\mathsf{Mod}$.
2. Let $\big((h'_k)_{k\in\mathbb{Z}}, (\partial'_k)_{k\in\mathbb{Z}}\big)$ be a homology theory on Top^2 with values in ${}_R\mathsf{Mod}$. Show that also $\big((h_k \oplus h'_k)_{k\in\mathbb{Z}}, (\partial_k \oplus \partial'_k)_{k\in\mathbb{Z}}\big)$ is a homology theory on Top^2 with values in ${}_R\mathsf{Mod}$.

Exercise 3 (homology of the torus)**.** Calculate the homology of the two-dimensional torus $T := S^1 \times S^1$ via the following strategy and illustrate your arguments in suitable way!:



Let $U \subset S^1$ be an open neighbourhood of $1 \in S^1$ and let $S := S^1 \times U \subset T$.

1. Use the long exact sequence of the pair and a topological argument to prove the following: For all $k \in \mathbb{Z}$ the inclusions $(T,\emptyset) \hookrightarrow (T,S)$ and $S \hookrightarrow T$ induce an R-isomorphism $h_k(T) \cong h_k(S) \oplus h_k(T,S)$.
2. Use excision to show that for all $k \in \mathbb{Z}$ we have R-isomorphisms
$$h_k(T,S) \cong h_k\big(S^1 \times [0,1], S^1 \times \{0\} \sqcup S^1 \times \{1\}\big).$$
3. Use the long exact triple sequence and excision to express the homology $h_k\big(S^1 \times [0,1], S^1 \times \{0\} \sqcup S^1 \times \{1\}\big)$ for all $k \in \mathbb{Z}$ in terms of the homology of S^1 and $\bullet$.
4. How can thus the homology $(h_k(T))_{k\in\mathbb{Z}}$ of T be expressed in terms of the homology of $\bullet$? State this result also explicitly in the case that the homology theory is an ordinary homology theory with values in ${}_{\mathbb{Z}}\mathsf{Mod}$ and coefficients isomorphic to $\mathbb{Z}$.

Please turn over

Exercise 4 (reduced homology)**.** If X is a topological space and $k \in \mathbb{Z}$, then we define the *k-th reduced homology of X with respect to* $\bigl((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\bigr)$ by

$$\widetilde{h}_k(X) := \ker\bigl(h_X(c_X)\colon h_k(X) \to h_k(\bullet)\bigr) \subset h_k(X),$$

where $c_X\colon X \longrightarrow \bullet$ is the uniquely determined map.

1. Show that for all topological spaces X, for all $x_0 \in X$, and for all $k \in \mathbb{Z}$ the composition

   $$\widetilde{h}_k(X) \longrightarrow h_k(X) \longrightarrow h_k(X, \{x_0\})$$

   induced by the inclusions is an R-isomorphism.

2. Show that for all continuous maps $f\colon X \longrightarrow Y$ of topological spaces and all $k \in \mathbb{Z}$ the R-homomorphism

   $$\widetilde{h}_k(f) := h_k(f)|_{\widetilde{h}_k(X)}\colon \widetilde{h}_k(X) \longrightarrow \widetilde{h}_k(Y)$$

   is well-defined and that this turns $\widetilde{h}_k$ into a homotopy invariant functor $\mathsf{Top} \longrightarrow {}_R\mathsf{Mod}$ with $\widetilde{h}_k(\bullet) \cong 0$.

3. State and prove an analogue of the long exact sequence of a pair in reduced homology.

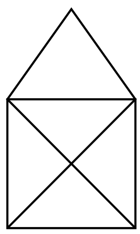
Bonus Problem (excision for homotopy groups?)**.**

1. Look up the statement of the *Blakers-Massey Theorem* in the literature.

2. Which consequences does it have for the calculation of homotopy groups of spheres?

3. How does the Blakers-Massey Theorem relate to the excision axiom for homology theories?

Nikolausaufgabe (4 bonus credits)**.** The traditional *Haus des Nikolaus* is the following subspace of $\mathbb{R}^2$:



1. In view of globalisation, Nikolaus has to expand his business. Therefore, given $n \in \mathbb{N}$, he wants to construct a path-connected n-sheeted covering space of the Haus des Nikolaus. Write a LaTeX macro \nikolaus with one argument such that \nikolaus{n} draws a beautiful path-connected n-sheeted covering of the Haus des Nikolaus. Execute \nikolaus{6}.

2. In order to be able to compete with Santa, Nikolaus eventually decides to invest into constructing the universal covering of the Haus des Nikolaus. Sketch this universal covering! Use colours!

Submission before December 11, 10:00, in the mailbox