Algebraic Topology I – Exercises

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Sheet 9, December 11, 2015

Hints. In the following, we write $\bullet := \{\emptyset\}$ for the canonical one-point space. Moreover, let R be a ring with unit and let $\big((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\big)$ be an ordinary homology theory on Top^2 with values in ${}_R\mathsf{Mod}$.

Exercise 1 (homology of coverings)**.** Let $p\colon Y \longrightarrow X$ be a covering map, let $R = \mathbb{Z}$ and let $h_0(\bullet) \cong \mathbb{Z}$. Prove or disprove:

1. For all $k \in \mathbb{Z}$ the induced map $h_k(p)\colon h_k(Y) \longrightarrow h_k(X)$ is surjective.
2. For all $k \in \mathbb{Z}$ the induced map $h_k(p)\colon h_k(Y) \longrightarrow h_k(X)$ is injective.

Exercise 2 (homology of bubbles)**.** Let $n, d \in \mathbb{N}_{>0}$. We denote the inclusions maps and collapse maps of the S^n-summands by $\big(i_j\colon S^n \to \bigvee^d S^n\big)_{j\in\{1,\dots,d\}}$ and $\big(p_j\colon \bigvee^d S^n \to S^n\big)_{j\in\{1,\dots,d\}}$ respectively. Prove that these maps induce for every $k \in \mathbb{N}_{>0}$ an isomorphism

$$h_k\Big(\bigvee^d S^n\Big) \cong \bigoplus^d h_k(S^n).$$

Exercise 3 (homology of the Klein bottle)**.** Let $R = \mathbb{Z}$ and let $Z := h_0(\bullet)$ be the coefficients of the ordinary homology theory $\big((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\big)$.

1. Calculate the homology of the Klein bottle in the case $Z \cong \mathbb{Z}$.
2. Calculate the homology of the Klein bottle in the case $Z \cong \mathbb{Z}/2$.

Exercise 4 (algebraic Mayer-Vietoris sequence)**.** Let

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{c_{k+1}} & A_k & \xrightarrow{a_k} & B_k & \xrightarrow{b_k} & C_k & \xrightarrow{c_k} & A_{k-1} & \xrightarrow{a_{k-1}} & \cdots \\
 & & \downarrow{\scriptstyle f_{A,k}} & & \downarrow{\scriptstyle f_{B,k}} & & \downarrow{\scriptstyle f_{C,k}} & & \downarrow{\scriptstyle f_{A,k-1}} & & \\
\cdots & \xrightarrow[c'_{k+1}]{} & A'_k & \xrightarrow[a'_k]{} & B'_k & \xrightarrow[b'_k]{} & C'_k & \xrightarrow[c'_k]{} & A'_{k-1} & \xrightarrow[a'_{k-1}]{} & \cdots
\end{array}$$

be a ($\mathbb{Z}$-indexed) commutative diagram in ${}_R\mathsf{Mod}$ with exact rows. Moreover, we assume that $f_{C,k}\colon C_k \longrightarrow C'_k$ is an isomorphism for all $k \in \mathbb{Z}$. For $k \in \mathbb{Z}$ we define

$$\Delta_k := c_k \circ f_{C,k}^{-1} \circ b'_k\colon B'k \longrightarrow A_{k-1}.$$

Prove via a diagram chase that then the sequence

$$\cdots \xrightarrow{\Delta_{k+1}} A_k \xrightarrow{(f_{A,k},\,-a_k)} A'_k \oplus B_k \xrightarrow{a'_k \oplus f_{B,k}} B'_k \xrightarrow{\Delta_k} A_{k-1} \longrightarrow \cdots$$

in ${}_R\mathsf{Mod}$ is exact.

Please turn over

Bonus Problem (realisation of homology groups)**.** Let $R = \mathbb{Z}$, let $h_0(\bullet) \cong \mathbb{Z}$, and let $k \in \mathbb{N}_{>0}$. Construct a functor

$$R_k \colon {}_{\mathbb{Z}}\mathsf{Mod} \longrightarrow \mathsf{Top}_{\mathrm{h}}$$

with $h_k \circ R_k \cong \mathrm{id}_{{}_{\mathbb{Z}}\mathsf{Mod}}$ and $h_\ell \circ R_k \cong 0$ for all $\ell \in \mathbb{N}_{>0} \setminus \{k\}$.
Hints. If A is an Abelian group, then there is a (natural) short exact sequence

$$0 \longrightarrow K(A) \longrightarrow F(A) \longrightarrow A \longrightarrow 0$$

in ${}_{\mathbb{Z}}\mathsf{Mod}$, where $F \colon \mathsf{Set} \longrightarrow {}_{\mathbb{Z}}\mathsf{Mod}$ is the free generation functor. Model this situation in Top via spheres and mapping cones ... If you want, you can restrict to the category of finitely generated Abelian groups.

Submission before December 18, 10:00, in the mailbox