Algebraic Topology – Exercises

Prof. Dr. C. Löh/D. Fauser/J. Witzig — Sheet 10, December 17, 2018

Hints. In the following, let $\bigl((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\bigr)$ be an ordinary homology theory on Top^2 with values in ${}_{\mathbb{Z}}\mathsf{Mod}$ and $h_0(\bullet) \cong_{\mathbb{Z}} \mathbb{Z}$.

Exercise 1 (injectivity/surjectivity on homology)**.** Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If $f\colon X \longrightarrow Y$ is an injective continuous map, then also the induced homomorphism $h_{2018}(f)\colon h_{2018}(X) \longrightarrow h_{2018}(Y)$ is injective.
2. If $f\colon X \longrightarrow Y$ is a surjective continuous map, then also the induced homomorphism $h_{2019}(f)\colon h_{2019}(X) \longrightarrow h_{2019}(Y)$ is surjective.

Exercise 2 (homology of the real projective plane/the Klein bottle)**.** Let K be the Klein bottle (Sheet 7, Exercise 2).

1. Compute $(h_k(\mathbb{R}P^2))_{k\in\mathbb{Z}}$ or $(h_k(K))_{k\in\mathbb{Z}}$.
2. What changes if the coefficients of $\bigl((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\bigr)$ are isomorphic to $\mathbb{Z}/2$ instead of $\mathbb{Z}$?

Illustrate your arguments in a suitable way!

Exercise 3 (homology vs. homotopy equivalence)**.** Give examples of topological spaces X and Y such that $h_k(X) \cong_{\mathbb{Z}} h_k(Y)$ holds for all $k \in \mathbb{Z}$, but $X \not\simeq Y$.

Exercise 4 (algebraic Mayer-Vietoris sequence)**.** Let R be a ring with unit and let

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{c_{k+1}} & A_k & \xrightarrow{a_k} & B_k & \xrightarrow{b_k} & C_k & \xrightarrow{c_k} & A_{k-1} & \xrightarrow{a_{k-1}} & \cdots \\
 & & \downarrow{\scriptstyle f_{A,k}} & & \downarrow{\scriptstyle f_{B,k}} & & \downarrow{\scriptstyle f_{C,k}} & & \downarrow{\scriptstyle f_{A,k-1}} & & \\
\cdots & \xrightarrow[c'_{k+1}]{} & A'_k & \xrightarrow[a'_k]{} & B'_k & \xrightarrow[b'_k]{} & C'_k & \xrightarrow[c'_k]{} & A'_{k-1} & \xrightarrow[a'_{k-1}]{} & \cdots
\end{array}$$

be a ($\mathbb{Z}$-indexed) commutative ladder in ${}_R\mathsf{Mod}$ with exact rows. Moreover, for every $k \in \mathbb{Z}$, let $f_{C,k}\colon C_k \longrightarrow C'_k$ be an isomorphism and let

$$\Delta_k := c_k \circ f_{C,k}^{-1} \circ b'_k \colon B'_k \longrightarrow A_{k-1}.$$

Show (via a diagram chase) that then the sequence

$$\cdots \xrightarrow{\Delta_{k+1}} A_k \xrightarrow{(f_{A,k},-a_k)} A'_k \oplus B_k \xrightarrow{a'_k \oplus f_{B,k}} B'_k \xrightarrow{\Delta_k} A_{k-1} \longrightarrow \cdots$$

in ${}_R\mathsf{Mod}$ is exact.

Bonus problem (realisation of homology groups)**.** Let $k \in \mathbb{N}_{>0}$. Construct a functor

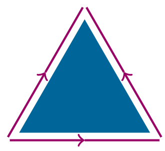
$$R_k \colon {}_{\mathbb{Z}}\mathsf{Mod}^{\mathsf{fin}} \longrightarrow \mathsf{Top_h}$$

with $h_k \circ R_k \cong \mathrm{Id}_{{}_{\mathbb{Z}}\mathsf{Mod}^{\mathsf{fin}}}$ and $h_\ell \circ R_k \cong 0$ for all $\ell \in \mathbb{N}_{>0} \setminus \{k\}$. Here, ${}_{\mathbb{Z}}\mathsf{Mod}^{\mathsf{fin}}$ denotes the category of all finitely generated $\mathbb{Z}$-modules.
Hints. Use spheres and mapping cones!

Please turn over

The following exercises will increase your Algebraic Topology XP!

Bonus problem (dunce cap/Narrenkappe)**.** The *dunce cap* is the quotient space of the (filled) triangle, obtained by identifying all three edges in the indicated way:



1. Try to draw the dunce cap!
2. Prove that the dunce cap is contractible.

   Hints. Mapping cones!

Bonus problem (trivial elements in homotopy groups)**.** Let $n \in \mathbb{N}$, let (X, x_0) be a pointed space and let $f \in \operatorname{map}_*((S^n, e_1^n), (X, x_0))$. Show that $[f]_*$ is trivial in $\pi_n(X, x_0)$ if and only if $f \colon S^n \longrightarrow X$ can be extended to a continuous map $D^{n+1} \longrightarrow X$. Illustrate!

Bonus problem ($\mathbb{R}P^\infty$)**.** We consider $S^\infty := \bigcup_{n \in \mathbb{N}} S^n$, where each sphere is included in the next one as equator; we endow S^∞ with the corresponding colimit topology. The *infinite real projective space* $\mathbb{R}P^\infty$ is then defined as the quotient of S^∞ by the corresponding antipodal action of $\mathbb{Z}/2$.

1. Calculate the fundamental group of $\mathbb{R}P^\infty$.
2. What can you say about the higher homotopy groups of $\mathbb{R}P^\infty$?

Bonus problem (the Heisenberg manifold)**.** We consider the *Heisenberg group*

$$H := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{Z} \right\} \subset H_\mathbb{R} := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\} \subset \mathrm{SL}_3(\mathbb{R})$$

and the *Heisenberg manifold* $M := H \setminus H_\mathbb{R}$, where H acts on $H_\mathbb{R}$ by matrix multiplication. Give detailed proofs for your answers to the following questions:

1. What is the fundamental group of M ?
2. Is there a covering of the 3-torus by M ?
3. Is there a covering of M by the 3-torus ?
4. Does M admit a path-connected regular covering whose deck transformation group is isomorphic to $\mathbb{Z}^2$?

Bonus problem (typos)**.** Find as many typos as possible in the lecture notes!

Submission before January 7, 2019, 10:00, in the mailbox

Merry Christmas and a Happy New Year!