

Algebraic Topology – Exercises

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Exercise 1 (chain homotopy equivalence). Let R be a principal ideal domain and let $C, D \in \text{Ob}({}_R\text{Ch})$. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If C is finite and $C \simeq_{R\text{Ch}} D$, then also D is finite.
2. If C and D are finite with $C \simeq_{R\text{Ch}} D$, then $\chi(C) = \chi(D)$.

Hints. Finite chain complexes and the algebraic Euler characteristic were introduced in Exercise 2 on Sheet 11.

Exercise 2 (diameter of affine simplices). Let $k \in \mathbb{N}$ and let $\sigma: \Delta^k \rightarrow \mathbb{R}^\infty$ be an affine linear simplex. Show that every summand τ in the definition of the barycentric subdivision $B_k(\sigma)$ satisfies

$$\text{diam}(\tau(\Delta^k)) \leq \frac{k}{k+1} \cdot \text{diam}(\sigma(\Delta^k)).$$

Hints. For $A \subset \mathbb{R}^\infty$, we write $\text{diam } A := \sup_{x,y \in A} \|x - y\|_2$. How can the diameter of affine linear simplices be expressed in terms of the vertices?

Exercise 3 (concrete cycles). Give an example of a singular cycle in $C_2(S^2; \mathbb{Z})$ that represents a non-trivial class in $H_2(S^2; \mathbb{Z})$ and prove that this cycle indeed has this property. Illustrate!

Exercise 4 (compatible homotopies). Let X be a topological space, let $(S_k \subset \text{map}(\Delta^k, X))_{k \in \mathbb{N}}$ be a family of simplices, and let $(h_\sigma)_{k \in \mathbb{N}, \sigma \in \text{map}(\Delta^k, X)}$ be a family of homotopies with the following properties:

1. For each $k \in \mathbb{N}$ and each $\sigma \in \text{map}(\Delta^k, X)$, the map $h_\sigma: \Delta^k \times [0, 1] \rightarrow X$ is a homotopy from σ to an element of S_k .
2. For all $k \in \mathbb{N}$, all $\sigma \in \text{map}(\Delta^k, X)$, and all $j \in \{0, \dots, k\}$, we have

$$h_{\sigma \circ i_{k,j}} = h_\sigma \circ (i_{k,j} \times \text{id}_{[0,1]}).$$

3. For all $k \in \mathbb{N}$ and all $\sigma \in S_k$, the homotopy h_σ satisfies

$$\forall x \in \Delta^k \quad \forall t \in [0, 1] \quad h_\sigma(x, t) = \sigma(x).$$

Let $C^S(X) \subset C(X)$ be the subcomplex of the singular chain complex generated in each degree $k \in \mathbb{N}$ by S_k instead of $\text{map}(\Delta^k, X)$.

Show that the inclusion $C^S(X) \rightarrow C(X)$ is a chain homotopy equivalence in $\mathbb{Z}\text{Ch}$ (and thus induces an isomorphism in homology).

Bonus problem (barycentric subdivision). Write a \LaTeX -macro that draws the barycentric subdivision of affine 2-simplices (specified by their vertices):



Bonus problem (Atiyah; additional bonus points!). Who was Sir Michael Atiyah? Why is this exercise posed right now? Formulate at least one theorem of Atiyah with topological relevance.