# Algebraic Topology – Exercises

Prof. Dr. C. Löh/D. Fauser/J. Witzig Sheet 13, January 21, 2019

**Exercise 1** (separation theorems?)**.** Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If $f \colon S^1 \longrightarrow S^1 \times S^1$ is continuous and injective, then $(S^1 \times S^1) \setminus f(S^1)$ has exactly two path-connected components.
2. If $f \colon S^1 \longrightarrow \mathbb{R}^{2019}$ is continuous and injective, then $\mathbb{R}^{2019} \setminus f(S^1)$ is path-connected.

**Exercise 2** (Slitherlink)**.** A *Slitherlink puzzle* consists of a square grid; some of the squares have numbers. The goal is to produce a closed loop out of the edges of the grid that is compatible with the given numbers in the following sense:

SL 1 Neighbouring grid points are joined by vertical or horizontal edges in such a way that we obtain a closed loop.

SL 2 The numbers indicate how many of the edges of a given square belong to the loop. For empty squares, the number of edges in the loop is not specified.

SL 3 The loop does not have any self-intersections or branches.

1. Solve the following Slitherlink puzzle:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | 3 | | | 3 | 2 | | | | | 2 |
| 2 | | 3 | | | 3 | | | | | 3 | |
| | | | 1 | | 1 | | | 1 | | | |
| | | 1 | | | | | 1 | 3 | 3 | 3 | 3 |
| | 3 | | 3 | 3 | 3 | 3 | | | 1 | | |
| 2 | | | | | | | 3 | | | | 1 |

2. How can the Jordan curve theorem be used to establish global strategies for solving Slitherlink puzzles? Give an example that illustrates this strategy.

**Exercise 3** (more on the $\ell^1$-semi-norm)**.** We consider the $\ell^1$-semi-norm on singular homology (Exercise 3 on Sheet 11).

1. Let $f \colon X \longrightarrow Y$ be a homotopy equivalence between topological spaces, let $k \in \mathbb{Z}$, and let $\alpha \in H_k(X; \mathbb{R})$. Show that
$$\big\| H_k(f; \mathbb{R})(\alpha) \big\|_1 = \|\alpha\|_1.$$
2. Let $n \in \mathbb{N}_{>0}$ and let $\alpha \in H_n(S^n; \mathbb{R})$. Show that $\|\alpha\|_1 = 0$.

*Please turn over*

**Exercise 4** (non-planarity of the torus)**.**

1. Let $n \in \mathbb{N}$, let $M$ be a compact, non-empty topological manifold of dimension $n$, let $N$ be a connected topological manifold of dimension $n$, and let $f \colon M \longrightarrow N$ be continuous and injective. Show that $f$ is surjective.

2. Conclude that there is *no* continuous injective map $S^1 \times S^1 \longrightarrow \mathbb{R}^2$.

**Bonus problem** (the five colour theorem)**.**

1. Choose a book on graph theory from the library that contains a proof of the five colour theorem.

2. Where does the proof use (relatives of) the Jordan curve theorem? Is this made explicit?

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Submission before January 28, 2019, 10:00, in the mailbox