

# Algebraic Topology – Exercises

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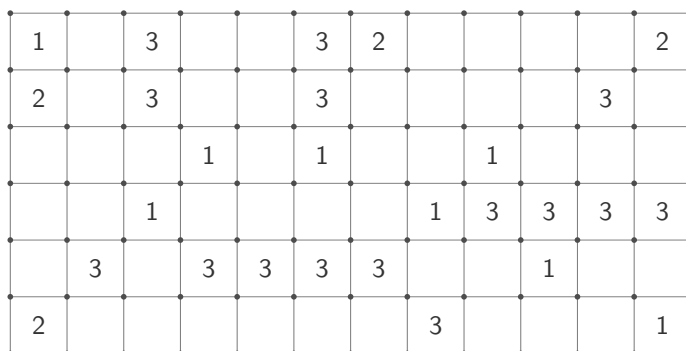
**Exercise 1** (separation theorems?). Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If  $f: S^1 \rightarrow S^1 \times S^1$  is continuous and injective, then  $(S^1 \times S^1) \setminus f(S^1)$  has exactly two path-connected components.
2. If  $f: S^1 \rightarrow \mathbb{R}^{2019}$  is continuous and injective, then  $\mathbb{R}^{2019} \setminus f(S^1)$  is path-connected.

**Exercise 2** (Slitherlink). A *Slitherlink* puzzle consists of a square grid; some of the squares have numbers. The goal is to produce a closed loop out of the edges of the grid that is compatible with the given numbers in the following sense:

- SL 1 Neighbouring grid points are joined by vertical or horizontal edges in such a way that we obtain a closed loop.
- SL 2 The numbers indicate how many of the edges of a given square belong to the loop. For empty squares, the number of edges in the loop is not specified.
- SL 3 The loop does not have any self-intersections or branches.

1. Solve the following Slitherlink puzzle:



2. How can the Jordan curve theorem be used to establish global strategies for solving Slitherlink puzzles? Give an example that illustrates this strategy.

**Exercise 3** (more on the  $\ell^1$ -semi-norm). We consider the  $\ell^1$ -semi-norm on singular homology (Exercise 3 on Sheet 11).

1. Let  $f: X \rightarrow Y$  be a homotopy equivalence between topological spaces, let  $k \in \mathbb{Z}$ , and let  $\alpha \in H_k(X; \mathbb{R})$ . Show that

$$\|H_k(f; \mathbb{R})(\alpha)\|_1 = \|\alpha\|_1.$$

2. Let  $n \in \mathbb{N}_{>0}$  and let  $\alpha \in H_n(S^n; \mathbb{R})$ . Show that  $\|\alpha\|_1 = 0$ .

*Please turn over*

**Exercise 4** (non-planarity of the torus).

1. Let  $n \in \mathbb{N}$ , let  $M$  be a compact, non-empty topological manifold of dimension  $n$ , let  $N$  be a connected topological manifold of dimension  $n$ , and let  $f: M \rightarrow N$  be continuous and injective. Show that  $f$  is surjective.
2. Conclude that there is *no* continuous injective map  $S^1 \times S^1 \rightarrow \mathbb{R}^2$ .

**Bonus problem** (the five colour theorem).

1. Choose a book on graph theory from the library that contains a proof of the five colour theorem.
2. Where does the proof use (relatives of) the Jordan curve theorem? Is this made explicit?