Algebraic Topology – Exercises

Prof. Dr. C. Löh/D. Fauser/J. Witzig Sheet 14, January 28, 2019

Exercise 1 (CW-spheres). Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. There exists a finite CW-complex X of dimension 2019 with $X \simeq_{\mathsf{Top}} S^1$.
2. There exists a finite CW-complex X of dimension 1 with $X \simeq_{\mathsf{Top}} S^{2019}$.

Exercise 2 (existence of finite coverings). Let X be a path-connected, locally path-connected, semi-locally simply connected, non-empty topological space such that $H_1(X;\mathbb{Z})$ is a finitely generated $\mathbb{Z}$-module with $\operatorname{rk}_{\mathbb{Z}} H_1(X;\mathbb{Z}) \geq 1$. Show that for every $d \in \mathbb{N}_{>0}$, there exists a connected d-sheeted covering of X.

Exercise 3 (homotopy vs. homology).

1. Let (X, x_0) be a simply connected pointed space and let $n \in \mathbb{N}_{>1}$ with

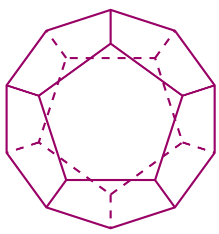
   $$\forall_{k\in\{1,\dots,n-1\}} \quad H_k(X;\mathbb{Z}) \cong 0.$$

   Using the Hurewicz theorem, show that then X is $(n-1)$-connected and the Hurewicz homomorphism $h_{(X,x_0),n} \colon \pi_n(X, x_0) \longrightarrow H_n(X;\mathbb{Z})$ is an isomorphism.
2. Compute $H_k(S^{2019} \times S^{2018};\mathbb{Z})$ for all $k \in \{0,\dots,2018\}$.

Exercise 4 (platonic solids).

1. How can the platonic solids be viewed as CW-structures on S^2 and D^3 ?
2. For each of these "platonic" CW-structures on S^2, compute

   $$\text{number of 0-cells} - \text{number of 1-cells} + \text{number of 2-cells}.$$



Bonus problem (Poincaré conjecture). What is the statement of the Poincaré conjecture? When have which cases been proved and by whom?

Submission before February 4, 10:00, in the mailbox