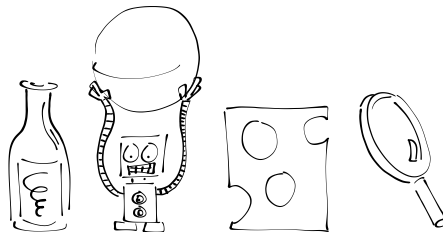# Algebraic Topology – Exercises

**The case of Lord Polygoto.** Detective Blorx is called to investigate the murder of Lord Polygoto at the Polygoto Mansion. Blorx quickly figures out that there are only four possible suspects (in alphabetic order):

- the butler

- the cook

- the gardener

- the librarian

Thorough questioning of the suspects revealed that, at the time of the murder, each of the suspects handled exactly one of the four suspicious items (A).

**A. The four suspicious items.**

- A Klein bottle.

- An ugly trophy made of steel (as depicted; the sphere on top is hollow, the statue is massive).

- A poisoned slice of Swiss cheese (as depicted).

- A magnifying lens (see schematics (D)).



**B. Statements by witnesses.**

- Drinking from the Klein bottle leads to a severe loss of orientation.

- The librarian or the gardener used the suspicious item with the maximal Euler characteristic.

- The gardener admires commutativity, especially in $\pi_1$. He would never touch anything non-commutative.

- The cook experimented with Betti spices and therefore handled an item such that $H_2( \, \cdot \, ; \mathbb{Q})$ and $H_3( \, \cdot \, ; \mathbb{Q})$ have different $\mathbb{Q}$-dimension.

- The butler never loses his way.

*Please turn over*

**C. The coroner's report.**

- The murder weapon admits a connected two-sheeted covering.

- The murder weapon has cyclic $H_1(\,\cdot\,;\mathbb{Z})$.

**D. Construction schematics of the magnifying lens.**

- Let $G \subset \mathbb{C}^\times$ be the subgroup of the multiplicative group $\mathbb{C}^\times$ consisting of all 2019-th roots of unity.

- View $S^3$ as unit sphere in $\mathbb{C}^2$.

- Let $\mathbb{C}^\times$ act on $S^3$ by scalar multiplication in $\mathbb{C}^2$.

- The lens is then the quotient space $G \setminus S^3$.

- 80/SK 340 H361 contains further information on the assembly of lenses.

**Exercise** (16 credits). Using the information collected in (A)–(D), help Blorx to answer the following questions:

1. Which items were the murder weapon(s)?

2. Who did it? More generally: Which suspect used which item?

*Hints.* The *Euler characteristic* of a finite CW-complex $X$ is defined as the alternating sum

$$\chi(X) := \sum_{n \in \mathbb{N}} (-1)^n \cdot \text{number of open } n\text{-cells of } X.$$

You may use that the Euler characteristic is homotopy invariant (among finite CW-complexes).

**Bonus problem** (pasta!). During the investigation, Blorx discovered that the cook and the librarian secretly worked on a joint book project *Topologhetti*: A dictionary between pasta and (algebraic) topology. Can you imagine how such a dictionary could look like?
*Hints.* Spaghetti, bucatini, fusilli, tortiglioni, penne rigate, tortellini, ravioli, lasagne, farfalle, ...

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