Algebraic Topology – Exercises

Prof. Dr. C. Löh/D. Fauser/J. Witzig Sheet 2, October 22, 2018

Exercise 1 (relative topology)**.** Let (X, A) and (Y, B) be pairs of topological spaces. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

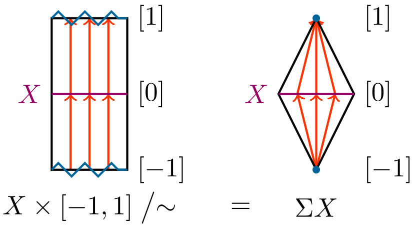
1. If $(X, A) \cong_{\mathsf{Top}^2} (Y, B)$, then $A \cong_{\mathsf{Top}} B$ and $X \cong_{\mathsf{Top}} Y$.
2. If $X/A \cong_{\mathsf{Top}} Y/B$, then $(X, A) \cong_{\mathsf{Top}^2} (Y, B)$.

Exercise 2 (suspension/Einhängung)**.** For a topological space X, we define the *suspension of* X as

$$\Sigma X := \bigl(X \times [-1, 1]\bigr) /{\sim}$$

(endowed with the quotient topology of the product toplogy), where "$\sim$" is the equivalence relation generated by

$$\forall_{x,x' \in X} \quad (x, 1) \sim (x', 1)$$
$$\forall_{x,x' \in X} \quad (x, -1) \sim (x', -1).$$



1. Let $f \colon X \longrightarrow Y$ be a continuous map. Show that the map Σf is well-defined and continuous:

$$\begin{aligned} \Sigma f \colon \Sigma X &\longrightarrow \Sigma Y \\ [x, t] &\longmapsto \bigl[f(x), t\bigr] \end{aligned}$$

2. Let $n \in \mathbb{N}$. Show that the following map is a well-defined homeomorphism:

$$\begin{aligned} \Sigma S^n &\longrightarrow S^{n+1} \\ [x, t] &\longmapsto \bigl(\cos(\pi/2 \cdot t) \cdot x, \sin(\pi/2 \cdot t)\bigr) \end{aligned}$$

Exercise 3 (projective plane via glueings)**.** Prove that there are pushout diagrams of topological spaces of the following type:

$$\begin{array}{ccc} S^0 & \longrightarrow & \mathbb{R}P^0 \\ \downarrow & & \downarrow \\ D^1 & \longrightarrow & \mathbb{R}P^1 \end{array} \qquad \begin{array}{ccc} S^1 & \longrightarrow & \mathbb{R}P^1 \\ \downarrow & & \downarrow \\ D^2 & \longrightarrow & \mathbb{R}P^2 \end{array}$$

In particular, describe all of the maps in these diagrams explicitly and illustrate your arguments graphically.

Please turn over

Exercise 4 (morphisms in the simplex category)**.** For $n \in \mathbb{N}_{>0}$ and $j \in \{0, \dots, n\}$ we define

$$\begin{aligned} d_j^n \colon \Delta(n-1) &\longrightarrow \Delta(n) \\ k &\longmapsto \begin{cases} k & \text{if } k < j \\ k+1 & \text{if } k \geq j; \end{cases} \end{aligned}$$

for $n \in \mathbb{N}$ and $j \in \{0, \dots, n\}$ we define

$$\begin{aligned} s_j^n \colon \Delta(n+1) &\longrightarrow \Delta(n) \\ k &\longmapsto \begin{cases} k & \text{if } k \leq j \\ k-1 & \text{if } k > j. \end{cases} \end{aligned}$$

Clearly, all of these maps are morphisms in the simplex category Δ.

1. Prove that every morphism in Δ is a composition of finitely many of the morphisms above.
2. Let $n \in \mathbb{N}$. Prove that for all $j, k \in \{0, \dots, n+1\}$ with $j < k$ we have

$$d_k^{n+1} \circ d_j^n = d_j^{n+1} \circ d_{k-1}^n.$$

Bonus problem (Asteroids)**.**

1. Explain how one could design a version of the computer game classic Asteroids on $\mathbb{R}P^2$ (instead of the 2-torus). Of course, screens are still assumed to be of rectangular shape; thus, it is necessary to first explain how to glue a rectangle into $\mathbb{R}P^2$. Moreover, special attention should be paid to the specification of controls (there is a subtlety that one needs to overcome).
2. Implement $\mathbb{R}P^2$-Asteroids. If you deploy it as a web application, then all participants of the course can enjoy it ...

Submission before October 29, 2018, 10:00, in the mailbox