

Algebraic Topology – Exercises

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Sheet 4, November 5, 2018

Exercise 1 (spheres and π_n). Let $n \in \mathbb{N}$. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If $\pi_n(S^n, e_1)$ consists of a single element, then for every pointed topological space (X, x_0) also $\pi_n(X, x_0)$ consists of a single element.
2. If $\pi_n(S^n, e_1)$ is infinite, then for every pointed space (X, x_0) also $\pi_n(S^n \times X, (e_1, x_0))$ is infinite.

Exercise 2 (product in Top_h). Let $(X_i)_{i \in I}$ be a family of topological spaces. Show that $\prod_{i \in I} X_i$ together with the homotopy classes of the canonical projections onto the factors satisfies the universal property of the product of $(X_i)_{i \in I}$ in the homotopy category Top_h .

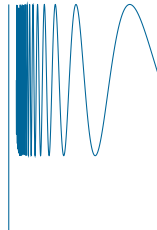
Exercise 3 (invariance of the boundary). In this exercise, you may assume that the theorem on existence of “interesting” homotopy invariant functors holds.

1. Let $n \in \mathbb{N}_{>0}$ and let $H^n := \{x \in \mathbb{R}^n \mid x_n \geq 0\}$ be the upper half-space (with respect to the subspace topology of \mathbb{R}^n). Prove *invariance of the boundary*, i.e., show that there is no open neighbourhood of 0 in H^n that is homeomorphic to the open unit ball $(D^n)^\circ$ in \mathbb{R}^n .

Hints. As first step, prove the following: If $U \subset H^n$ is an open neighbourhood of 0 in H^n , then $U \setminus \{0\} \simeq U$.

2. Conclude that the Möbius strip and the boring strip $S^1 \times [0, 1]$ are *not* homeomorphic.

Exercise 4 (Warsaw circle/Warschauer Kreis). The topological space



$$W := \{(x, \sin(2 \cdot \pi/x)) \mid x \in (0, 1]\} \\ \cup (\{1\} \times [-2, 0]) \cup ([0, 1] \times \{-2\}) \cup (\{0\} \times [-2, 1])$$

(endowed with the subspace topology of \mathbb{R}^2) is called *Warsaw circle*. Prove that for every baspoint $w_0 \in W$ the fundamental group $\pi_1(W, w_0)$ is trivial.

Hints. Show first that no loop in W can cross the “gap.”

Bonus problem (Nash equilibria). Look up the notion of Nash equilibria in Game Theory. Prove the existence of Nash equilibria using the Brouwer fixed point theorem.

Submission before November 12, 2018, 10:00, in the mailbox