Algebraic Topology – Exercises

Prof. Dr. C. Löh/D. Fauser/J. Witzig — Sheet 5, November 12, 2018

Hints. You may use that the following map is a group isomorphism:

$$\begin{aligned} \mathbb{Z} &\longrightarrow \pi_1(S^1, e_1) \\ d &\longmapsto \bigl[[t] \mapsto [d \cdot t \mod 1]\bigr]_* \end{aligned}$$

Exercise 1 (injectivity/surjectivity and π_1)**.** Let $f\colon (X, x_0) \longrightarrow (Y, y_0)$ be a pointed continuous map between pointed spaces. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If f is injective, then $\pi_1(f)\colon \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$ is injective.
2. If f is surjective, then $\pi_1(f)\colon \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$ is surjective.

Exercise 2 (π_0 and path-connected components)**.** Look up the definition of path-connected components of topological spaces. Let (X, x_0) be a pointed space and let $\operatorname{PC}(X)$ be the set of path-connected components of X. Prove that the following map is a well-defined bijection:

$$\begin{aligned} \pi_0(X, x_0) &\longrightarrow \operatorname{PC}(X) \\ [\gamma]_* &\longmapsto [\gamma(-1)] \end{aligned}$$

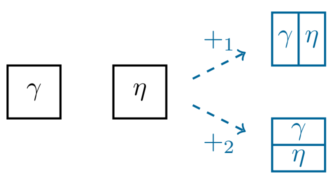
Exercise 3 (π_1 and contractibility)**.** Let X be a topological space.

1. Let $\gamma\colon S^1 \longrightarrow X$ be a null-homotopic map and let $x_0 := \gamma(1)$. Show that then $[\gamma]_*$ is trivial in $\pi_1(X, x_0)$. Illustrate your argument in a suitable way!
2. Conclude: If X is contractible (but not necessarily pointedly contractible!) and $x_0 \in X$, then $\pi_1(X, x_0)$ is the trivial group.

Exercise 4 (group structure on π_n)**.** Let $n \in \mathbb{N}_{\geq 2}$ and let the composition maps $+_1, \dots, +_n\colon \pi_n(X, x_0) \times \pi_n(X, x_0) \longrightarrow \pi_n(X, x_0)$ be defined as in Outlook 2.1.6.

1. Prove that $+_j = +_1$ for all $j \in \{1, \dots, n\}$.
2. Prove that $\pi_n(X, x_0)$ is an Abelian group with respect to $+_1$.

Hints. Eckmann and Hilton might help!



Bonus problem (fundamental theorem of algebra)**.** Use $\pi_1(S^1, e_1)$ to prove that every non-constant polynomial in $\mathbb{C}[X]$ has at least one root in $\mathbb{C}$.
Hints. Show that a non-constant polynomial $p \in \mathbb{C}[X]$ without roots in $\mathbb{C}$ would yield a map $S^1 \longrightarrow S^1$ that is both null-homotopic and homotopic to the map $S^1 \longrightarrow S^1$, $[t] \mapsto [\deg p \cdot t \mod 1]$, ...

Submission before November 19, 2018, 10:00, in the mailbox