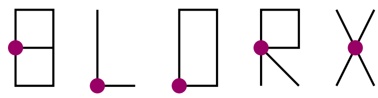
Algebraic Topology – Exercises

Prof. Dr. C. Löh/D. Fauser/J. Witzig Sheet 6, November 19, 2018

Hints. You may use that the following map is a group isomorphism:

$$\begin{aligned} \mathbb{Z} &\longrightarrow \pi_1(S^1, e_1) \\ d &\longmapsto \bigl[[t] \mapsto [d \cdot t \mod 1]\bigr]_* \end{aligned}$$

Exercise 1 (BLORX!). We consider the following five subspaces of $\mathbb{R}^2$, with the indicated basepoints:



Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. These spaces consist of exactly five homeomorphism types.
2. These spaces consist of exactly four pointed homotopy types.

Exercise 2 (pushouts of groups). We consider pushout diagrams in Group of the following type:

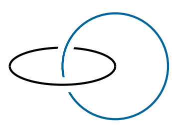
$$\begin{array}{ccc} 1 & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & F \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \downarrow \\ C & \longrightarrow & G \end{array}$$

(where A, B, C, F, G are groups, $f\colon A \longrightarrow B$ is a group homomorphism, and $g\colon A \longrightarrow C$ is a group isomorphism). Prove the following statements via the universal property of pushouts:

1. The group F is *not* Abelian.
2. The group G is isomorphic to B.

Exercise 3 (fundamental group of the real projective plane). Use the theorem of Seifert and van Kampen to prove that $\pi_1(\mathbb{R}P^2, [e_1]) \cong \mathbb{Z}/2$ and describe a generating loop explicitly.

Exercise 4 (Houdini?). A rubber ring and a steel ring are entangled in $\mathbb{R}^3$ as illustrated below. Moreover, you may assume that the rubber ring is fixed at one point at all times.



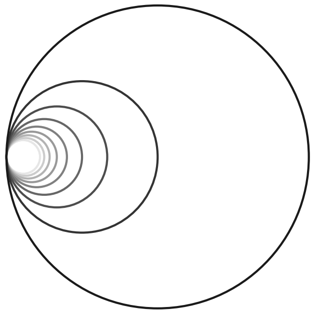
Can these two rings be unlinked in $\mathbb{R}^3$ by deforming the rubber ring (without cutting it)?

1. Model the situation above, using appropriate topological terms.
2. Answer the question above (with a full proof) with respect to your model.
3. What changes if we look at the same rings in $\mathbb{R}^4$ instead of $\mathbb{R}^3$?

Illustrate your arguments in a suitable way!

Please turn over

Bonus problem (Hawaiian earring). We consider the following subspace H of $\mathbb{R}^2$ with the subspace topology, the so-called *Hawaiian earring*:



$$H := \bigcup_{n\in\mathbb{N}_{>0}} \{x \in \mathbb{R}^2 \mid d(x, (1/n, 0)) = 1/n\}$$

Prove that $\pi_1(H, 0)$ is uncountable and illustrate your arguments in a suitable way. Is $(H, 0)$ (pointedly) homotopy equivalent to $\bigvee_{\mathbb{N}}(S^1, e_1)$?

Submission before November 26, 2018, 10:00, in the mailbox