Algebraic Topology – Exercises

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Hints. You may use that the following map is a group isomorphism:

$$\begin{aligned} \mathbb{Z} &\longrightarrow \pi_1(S^1, e_1) \\ d &\longmapsto \bigl[[t] \mapsto [d \cdot t \mod 1]\bigr]_* \end{aligned}$$

Exercise 1 (coverings)**.** Let X, Y, Z be topological spaces and let $p\colon X \longrightarrow Y$, $q\colon Y \longrightarrow Z$ be continuous maps. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If $q \circ p$ is a covering map, then also p is a covering map.
2. If $q \circ p$ is a covering map, then also q is a covering map.

Exercise 2 (Klein bottle/Kleinsche Flasche)**.** Let K be the *Klein bottle*, i.e., the quotient space $K = ([0,1] \times [0,1])/{\sim}$ defined by the following glueing relation:

1. Show that the fundamental group of K (at the basepoint $([0],[0])$) is non-Abelian.
2. Show that there exists a 2-sheeted covering $S^1 \times S^1 \longrightarrow K$. Draw it!

Exercise 3 (pretzel coverings)**.** Let $(B, b) := (S^1, e_1) \vee (S^1, e_1)$.

1. Construct (and draw) two connected 2-sheeted coverings of (B, b) that are *not* isomorphic in $\mathsf{Cov}_{(B,b)}$ (and prove that they are not isomorphic).
2. Construct (and draw) a connected 3-sheeted covering of (B, b) whose deck transformation group does *not* act transitively on the fibres (and prove this fact).

Exercise 4 (coverings of wild spaces)**.**

1. Show that the Warsaw circle (Exercise 4 of Sheet 4) admits a non-trivial covering. Draw it!

   Hints. Warsaw helix!

2. Show that the Hawaiian earring (Bonus problem of Sheet 6) admits *no* simply connected covering with non-empty total space.

Please turn over

Bonus problem (one-dimensional complexes). A *one-dimensional complex* is a pair (X, X_0), consisting of a topological space X and a discrete subspace X_0 with the following property: There exists a set I and a pushout (in Top) of the form

$$\begin{array}{ccc} \coprod_I S^0 & \longrightarrow & X_0 \\ \downarrow & & \downarrow \\ \coprod_I D^1 & \longrightarrow & X \end{array}$$

where the left vertical arrow is the canonical inclusion and the right vertical arrow is the inclusion of X_0 into X. I.e., one-dimensional complexes can be obtained by glueing intervals at their end-points in a certain way.

1. Let (X, X_0) be a one-dimensional complex and let $x_0 \in X_0$. Show that $\pi_1(X, x_0)$ is a free group.

   Hints. If you want, you can restrict to the case that X_0 and the set I are finite.

2. Let (X, X_0) be a one-dimensional complex and let $p\colon Y \longrightarrow X$ be a covering map. Show that then also $(Y, p^{-1}(X_0))$ is a one-dimensional complex.

Submission before December 3, 2018, 10:00, in the mailbox