Algebraic Topology – Exercises

Prof. Dr. C. Löh/D. Fauser/J. Witzig Sheet 8, December 3, 2018

Exercise 1 (classification of coverings)**.** Let $X := S^{2017} \times \mathbb{R}P^{2018} \times \mathbb{R}P^{2019}$. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

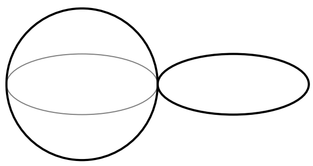
1. There exists a path-connected covering of X with infinitely many sheets.
2. Every path-connected finite covering of X is regular.

Exercise 2 (The Borsuk-Ulam theorem in dimension 2)**.** A map $f\colon S^2 \longrightarrow S^1$ is *antipodal* if $f(-x) = -f(x)$ holds for all $x \in S^2$.

1. Show that there is no continuous antipodal map $S^2 \longrightarrow S^1$.

   Hints. Try to use $\mathbb{R}P^2$ and consider a path in S^2 from e_1 to $-e_1$.
2. Prove the *Borsuk-Ulam theorem in dimension* 2: If $f\colon S^2 \longrightarrow \mathbb{R}^2$ is continuous, then there exists $x \in S^2$ with $f(x) = f(-x)$.

Exercise 3 (large homotopy groups)**.** Let $(X, x_0) := (S^2, e_1) \vee (S^1, e_1)$.



1. Give a simple geometric description of the universal covering of X and prove that this indeed is a universal covering of X.
2. Let $k \in \mathbb{N}_{\geq 2}$ with the property that $\pi_k(S^2, e_1^2)$ is non-trivial. Prove that then $\pi_k(X, x_0)$ is *not* finitely generated (as Abelian group).

Exercise 4 (no finite coverings)**.** Show that there exists a path-connected, locally path-connected, semi-locally simply connected, non-empty topological space that is *not* simply connected and does *not* admit a finite-sheeted non-trivial covering.
Hints. Don't be irrational!

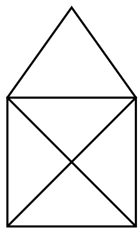
Bonus problem (isometry groups)**.** Let (M, g) be a compact connected Riemannian manifold and let $p\colon \widetilde{M} \longrightarrow M$ be its universal covering (with the smooth structure induced by M). Let $\widetilde{g} := p^* g$ be the induced Riemannian metric on $\widetilde{M}$. Moreover, let $x_0 \in M$.

1. Prove that the deck transformation action of $\pi_1(X, x_0)$ (with respect to some chosen point in $p^{-1}(x_0)$) on $(\widetilde{M}, \widetilde{g})$ is isometric.
2. Prove that this defines an embedding of $\pi_1(X, x_0)$ into the isometry group of $(\widetilde{M}, \widetilde{g})$ as a uniform lattice.

   Hints. Here, we equip the isometry group with the compact-open topology. A subgroup of a topological group is a *uniform lattice* (uniformes Gitter) if this subgroup is discrete (with respect to the subspace topology) and the quotient space (with respect to the quotient topology) is compact.

Please turn over

Nikolausaufgabe (additional 4 bonus credits)**.** The traditional *Haus des Nikolaus* is the following subspace of $\mathbb{R}^2$:



The new edition of this house will be built by the Blorx Building Trust (which won this contract in a transparent, corruption-free procedure).

1. Given $n \in \mathbb{N}$, the Blorx Building Trust will construct a path-connected n-sheeted covering space of the Haus des Nikolaus. Write a LaTeX macro \nikolaus with one argument such that \nikolaus{n} draws a beautiful path-connected n-sheeted covering of the Haus des Nikolaus. Execute \nikolaus{8}.

2. In order to be able to compete with Santa, Nikolaus eventually mandates the Blorx Building Trust to construct the universal covering of the Haus des Nikolaus. Sketch this universal covering! Use colours!

Submission before December 10, 2018, 10:00, in the mailbox