Algebraic Topology – Exercises

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Hints. In the following, let $\bullet = \{\emptyset\}$ denote “the” one-point space, let R be a ring with unit and let $\big((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\big)$ be a homology theory on Top^2 with values in ${}_R\mathsf{Mod}$.

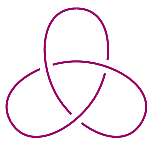
Exercise 1 (more homology theories?)**.** Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. Then also $((h_{k+2018})_{k\in\mathbb{Z}}, (\partial_{k+2018})_{k\in\mathbb{Z}})$ is a homology theory on Top^2 with values in ${}_R\mathsf{Mod}$.
2. If R is an integral domain and Q is the quotient field of R, then also the pair $((Q \otimes_R h_k)_{k\in\mathbb{Z}}, (\mathrm{id}_Q \otimes_R \partial_k)_{k\in\mathbb{Z}})$ is a homology theory on Top^2 with values in ${}_Q\mathsf{Mod}$.

Exercise 2 (knots)**.** An *embedded knot* is a smooth embedding $S^1 \longrightarrow \mathbb{R}^3$. The *knot complement* of an embedded knot $K\colon S^1 \longrightarrow \mathbb{R}^3$ is then $\mathbb{R}^3 \setminus K(S^1)$.

1. Does it make sense to study embedded knots up to homotopy of maps?
2. Does it make sense to study embedded knots by considering homology of knot complements?

   Hints. You may use the following version of the *tubular neighbourhood theorem*: If K is an embedded knot, then there exists a compact subset $N \subset \mathbb{R}^3$ and a homeomorphism $f\colon S^1 \times D^2 \longrightarrow N$ with $f(S^1 \times \{0\}) = K(S^1)$.



Exercise 3 (homology of the torus)**.** Calculate the homology of the two-dimensional torus $T := S^1 \times S^1$ via the following strategy and illustrate your arguments in suitable way! Let $U \subset S^1$ be an appropriate open neighbourhood of $e_1 \in S^1$ and let $S := S^1 \times U \subset T$.

1. Use the long exact sequence of the pair and a topological argument to prove the following: For all $k \in \mathbb{Z}$ the inclusions $(T, \emptyset) \hookrightarrow (T, S)$ and $S \hookrightarrow T$ induce an R-isomorphism $h_k(T) \cong h_k(S) \oplus h_k(T, S)$.
2. Use excision to show that for all $k \in \mathbb{Z}$ we have R-isomorphisms
$$h_k(T, S) \cong h_k\big(S^1 \times [0,1], S^1 \times \{0,1\}\big).$$
3. Use the long exact triple sequence and excision to express the homology $h_k\big(S^1 \times [0,1], S^1 \times \{0,1\}\big)$ for all $k \in \mathbb{Z}$ in terms of the homology of S^1.
4. Compare this result with the homotopy groups of $(T, (e_1, e_1))$.

Please turn over

Exercise 4 (reduced homology)**.** If X is a topological space and $k \in \mathbb{Z}$, then we define the *k-th reduced homology of X with respect to* $\bigl((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\bigr)$ by

$$\widetilde{h}_k(X) := \ker\bigl(h_X(c_X)\colon h_k(X) \to h_k(\bullet)\bigr) \subset h_k(X),$$

where $c_X \colon X \longrightarrow \bullet$ is the constant map.

1. Show that for all topological spaces X, for all $x_0 \in X$, and for all $k \in \mathbb{Z}$ the composition

$$\widetilde{h}_k(X) \longrightarrow h_k(X) \longrightarrow h_k(X, \{x_0\})$$

   of the inclusion and the homomorphism induced by the inclusion is an R-isomorphism.

2. Show that for all continuous maps $f \colon X \longrightarrow Y$ of topological spaces and all $k \in \mathbb{Z}$ the R-homomorphism

$$\widetilde{h}_k(f) := h_k(f)|_{\widetilde{h}_k(X)} \colon \widetilde{h}_k(X) \longrightarrow \widetilde{h}_k(Y)$$

   is well-defined and that this turns $\widetilde{h}_k$ into a homotopy invariant functor $\mathsf{Top} \longrightarrow {}_R\mathsf{Mod}$ with $\widetilde{h}_k(\bullet) \cong 0$.

Bonus problem (Abelian categories)**.**

1. Look up the term *Abelian category* in the literature.
2. Give an example for an additive category that is *not* Abelian.
3. How can exact sequences be defined in Abelian categories?
4. What does the Freyd-Mitchell embedding theorem say?

Submission before December 17, 2018, 10:00, in the mailbox