Algebraic Topology – Etudes

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Let $\big((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\big)$ be a homology theory on Top^2 with values in ${}_{\mathbb{Z}}\mathsf{Mod}$ and $h_0(\bullet) \cong_{\mathbb{Z}} \mathbb{Z}$ [alternatively: $h_0(\bullet) \cong_{\mathbb{Z}} \mathbb{Z}/2$ or $h_0(\bullet) \cong_{\mathbb{Z}} \mathbb{Q}$].

Exercise 1 (Mayer-Vietoris). Let $k \in \mathbb{Z}$.

1. Compute $h_k(S^2)$ via a suitable Mayer-Vietoris sequence from $h_k(S^1)$.
2. Compute $h_k(S^1 \times S^1)$ via a Mayer-Vietoris sequence associated with the decomposition of the torus $S^1 \times S^1$ into two cylinders.

Exercise 2 (mapping cones). Let

$$\begin{aligned} f_3 \colon S^1 &\longrightarrow S^1 \\ [t] &\longmapsto [3 \cdot t \mod 1] \end{aligned}$$

and let $k \in \mathbb{Z}$.

1. Compute $h_k(\operatorname{Cone}(f_3))$.
2. Compute $h_k(\Sigma \operatorname{Cone}(f_3))$.

Exercise 3 (chain complexes and their homology). Recall the notion of chain complexes, chain maps, and their homology (Appendix A.6.2):

1. What is the definition of chain complexes and chain maps?
2. What are typical examples?
3. What is the homology of a chain complex?
4. How can homology be computed?
5. How does all this relate to exactness?
6. Why did we introduce chain complexes in Commutative Algebra?

Exercise 4 (summary). Write a summary of Chapter 3.2 (Homology of Spheres and Suspensions) and Chapter 3.3 (Glueings: The Mayer-Vietoris Sequence), keeping the following questions in mind:

1. How can the homology of spheres/suspensions be computed?
2. What can you say about mapping degrees for self-maps of spheres (in ordinary homology)?
3. How can the Mayer-Vietoris sequence be used to compute homology of glueings?
4. What is the mapping cone trick?
5. How can relative homology be viewed as absolute homology?

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no submission!