Algebraic Topology – Etudes

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Exercise 1 (concatenation of loops)**.** Let $\gamma := \mathrm{id}_{S^1} \colon (S^1, e_1) \longrightarrow (S^1, e_1)$. Draw pictures of the following loops:

$$\overline{\gamma}, \qquad \gamma * \gamma, \qquad \gamma * (\gamma * \gamma), \qquad \gamma * \overline{\gamma}, \qquad (\gamma * \overline{\gamma}) * \gamma$$

Exercise 2 (loops in spheres)**.**

1. Let $\gamma \colon (S^1, e_1) \longrightarrow (S^2, e_1)$ be the inclusion as equator. Show that $[\gamma]_*$ is trivial in $\pi_1(S^2, e_1)$.
2. Let
$$\begin{aligned} \gamma \colon (S^1, e_1) &\longrightarrow \big(S^1 \times S^1, (e_1, e_1)\big) \\ [t] &\longmapsto \big([t], [2 \cdot t]\big), \end{aligned}$$
and let $p_1, p_2 \colon S^1 \times S^1 \longrightarrow S^1$ be the canonical projections. Compute $\pi_1(p_1)([\gamma]_*)$ and $\pi_1(p_2)([\gamma]_*)$ and draw the corresponding pictures!

Exercise 3 (pushouts)**.**

1. Let X be a topological space, let X_1, X_2 be subspaces of X with $X = X_1^\circ \cup X_2^\circ$, and let $X_0 := X_1 \cap X_2$. Show that X (together with the inclusions) is a pushout in Top of the diagram
$$\begin{array}{ccc} X_0 & \xrightarrow{\text{inclusion}} & X_2 \\ {\scriptstyle \text{inclusion}}\downarrow & & \\ X_1 & & \end{array}$$
2. Let $X_1 := S^2 \setminus \{e_2\}$, $X_2 := S^2 \setminus \{-e_2\}$ and $X_0 := X_1 \cap X_2$. Is (S^2, e_1) a pushout of
$$\begin{array}{ccc} (X_0, e_1) & \xrightarrow{[\text{inclusion}]_*} & (X_2, e_1) \\ {\scriptstyle [\text{inclusion}]_*}\downarrow & & \\ (X_1, e_1) & & \end{array}$$
in $\mathsf{Top}_{*\mathsf{h}}$ (provided that (S^2, e_1) is *not* pointedly contractible)?!

Exercise 4 (summary)**.** Write a summary of Chapter 2.1 (The Fundamental Group), keeping the following questions in mind:

1. How are the functors π_n defined?
2. What is the geometric idea behind π_n ?
3. How is the group structure on π_1 defined?
4. Why/How is the basepoint relevant?

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no submission!