Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold/J. Witzig Sheet 1, October 19, 2021

Exercise 1 (product topology)**.** Let X and Y be topological spaces. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If $B \subset X$ and $C \subset Y$ are closed subsets, then $B \times C \subset X \times Y$ is closed.
2. If $A \subset X \times Y$ is closed, then there are closed sets $B \subset X$ and $C \subset Y$ with $A = B \times C$.

Exercise 2 (TOPOLOGY)**.** Classify the following six subspaces of $\mathbb{R}^2$ up to homeomorphism and prove this classification result.

Hints. Some of the homeomorphisms might be hard to write down explicitly; in these cases, it is sufficient to give an outline on how to construct them and to indicate clearly that a proper formal argument would require more details.

Exercise 3 (stereographic projection)**.** Let $n \in \mathbb{N}_{>0}$ and $N := (0, \dots, 0, 1) \in S^n$; i.e., N is the North Pole of S^n. The map

$$\begin{aligned} s_n \colon S^n \setminus \{N\} &\longrightarrow \mathbb{R}^n \\ (x_1, \dots, x_{n+1}) &\longmapsto \frac{1}{1 - x_{n+1}} \cdot (x_1, \dots, x_n) \end{aligned}$$

is called *stereographic projection*. Give a geometric interpretation of this map and prove that it is a homeomorphism. Illustrate your arguments graphically!

Exercise 4 (balls, spheres, simplices)**.** Let $n \in \mathbb{N}_{>0}$. Solve one of the following problems:

1. Prove that Δ^n is homeomorphic to D^n and that $\partial \Delta^n$ is homeomorphic to S^{n-1}.

   Hints. Find the centre and inflate!

2. Prove that D^n / S^{n-1} is homeomorphic to S^n.

   Hints. Quotient spaces will be introduced in the second lecture.

Illustrate your arguments graphically!
Hints. The compact-Hausdorff trick might be useful.

Bonus problem (Peano curves)**.** Show that there exist surjective continuous maps $[0, 1] \longrightarrow [0, 1] \times [0, 1]$. Can such a map be injective?

Submission before October 26, 2021, 8:30, via GRIPS

(Solutions may be submitted in English or German.)