Algebraic Topology: Exercises

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Hints. Let $\bullet = \{\emptyset\}$ denote "the" one-point space, let R be a ring with unit and let $\bigl((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\bigr)$ be a homology theory on Top^2 with values in ${}_R\mathsf{Mod}$.

Exercise 1 (more homology theories?). Which of the following statements are true? Justify your answer with a suitable proof or counterexample (a brief explanation is enough!).

1. Then $((h_{k+2022})_{k\in\mathbb{Z}}, (\partial_{k+2022})_{k\in\mathbb{Z}})$ is a homology theory on Top^2 with values in ${}_R\mathsf{Mod}$.
2. If R is an integral domain and Q is the quotient field of R, then also the pair $((Q \otimes_R h_k)_{k\in\mathbb{Z}}, (\mathrm{id}_Q \otimes_R \partial_k)_{k\in\mathbb{Z}})$ is a homology theory on Top^2 with values in ${}_Q\mathsf{Mod}$.

Exercise 2 (reduced homology). If X is a topological space and $k \in \mathbb{Z}$, then we define the *k-th reduced homology of X with respect to* $\bigl((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\bigr)$ by

$$\widetilde{h}_k(X) := \ker\bigl(h_X(c_X) \colon h_k(X) \to h_k(\bullet)\bigr) \subset h_k(X),$$

where $c_X \colon X \longrightarrow \bullet$ is the constant map.

1. Show that for all topological spaces X, for all $x_0 \in X$, and for all $k \in \mathbb{Z}$ the composition

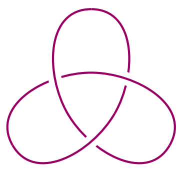
   $$\widetilde{h}_k(X) \longrightarrow h_k(X) \longrightarrow h_k(X, \{x_0\})$$

   of the inclusion and the homomorphism induced by the inclusion is an R-isomorphism.
2. Compute the reduced homology of contractible spaces.

Exercise 3 (knots). An *embedded knot* is a smooth embedding $S^1 \longrightarrow \mathbb{R}^3$. The *knot complement* of an embedded knot $K \colon S^1 \longrightarrow \mathbb{R}^3$ is $\mathbb{R}^3 \setminus K(S^1)$.

1. Does it make sense to study embedded knots up to homotopy of maps?
2. Does it make sense to study embedded knots by considering ordinary homology of knot complements?

   Hints. You may use the following *tubular neighbourhood theorem*: If K is an embedded knot, then there exists a compact subset $N \subset \mathbb{R}^3$ and a homeomorphism $f \colon S^1 \times D^2 \longrightarrow N$ with $f(S^1 \times \{0\}) = K(S^1)$.



Please turn over

Exercise 4 (homology of the torus)**.** Calculate the homology of the two-dimensional torus $T := S^1 \times S^1$ via the following strategy and illustrate your arguments in suitable way! Let $U \subset S^1$ be an appropriate open neighbourhood of $e_1 \in S^1$ and let $S := S^1 \times U \subset T$. Solve two of the following:

1. Use the long exact sequence of the pair and a topological argument to prove the following: For all $k \in \mathbb{Z}$ the inclusions $(T, \emptyset) \hookrightarrow (T, S)$ and $S \hookrightarrow T$ induce an R-isomorphism $h_k(T) \cong h_k(S) \oplus h_k(T, S)$.
2. Use excision to show that for all $k \in \mathbb{Z}$ we have R-isomorphisms
$$h_k(T, S) \cong h_k\bigl(S^1 \times [0,1], S^1 \times \{0,1\}\bigr).$$
3. Use the long exact triple sequence and excision to express the homology $h_k\bigl(S^1 \times [0,1], S^1 \times \{0,1\}\bigr)$ for all $k \in \mathbb{Z}$ in terms of the homology of S^1.

Bonus problem (tweet)**.** Write a tweet (no more than 140 characters) stating the Seifert and van Kampen theorem, without using mathematical symbols.

Bonus problem (poster)**.** Design a poster that advertises the classification theorem for coverings. Use colours!

Bonus problem (Klein tic-tac-toe)**.** Write instructions on how to play tic-tac-toe on a Klein bottle. Illustrate your instructions with pictures.

Bonus problem (poetry)**.**

And for the classically oriented mind:
don't leave the Brouwer fixed point theorem behind!
Theorem and proof need a formulation,
using rhymes as decoration.

Submission before January 11, 2022, 8:30, via GRIPS
Merry Christmas and a Happy New Year!