Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold/J. Witzig Sheet 11, January 11, 2022

Hints. In the following, let $\bigl((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\bigr)$ be an ordinary homology theory on Top^2 with values in ${}_{\mathbb{Z}}\mathsf{Mod}$ and $h_0(\bullet) \cong_{\mathbb{Z}} \mathbb{Z}$.

Exercise 1 (surjectivity on homology)**.** Which of the following statements are true for all continuous maps $f\colon X \longrightarrow Y$ of spaces? Justify your answer with a suitable proof or counterexample.

1. If f is surjective, then also $h_{2022}(f)\colon h_{2022}(X) \longrightarrow h_{2022}(Y)$ is surjective.
2. If $h_{2022}(f)\colon h_{2022}(X) \longrightarrow h_{2022}(Y)$ is surjective, then f is surjective.

Exercise 2 (homology of the real projective plane/the Klein bottle)**.** Let K be the Klein bottle (Sheet 7, Exercise 2).

1. Compute $(h_k(\mathbb{R}P^2))_{k\in\mathbb{Z}}$ or $(h_k(K))_{k\in\mathbb{Z}}$.
2. What changes if the coefficients of $\bigl((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\bigr)$ are isomorphic to $\mathbb{Z}/2$ instead of $\mathbb{Z}$?

Illustrate your arguments in a suitable way!

Exercise 3 (homology vs. homotopy equivalence)**.** Give examples of topological spaces X and Y such that $h_k(X) \cong_{\mathbb{Z}} h_k(Y)$ holds for all $k \in \mathbb{Z}$, but $X \not\simeq Y$.
Hints. Dimension 2 is sufficient.

Exercise 4 (algebraic Mayer-Vietoris sequence)**.** Let R be a ring with unit and let

$$\begin{tikzcd}
\cdots \arrow[r, "c_{k+1}"] & A_k \arrow[r, "a_k"] \arrow[d, "f_{A,k}"'] & B_k \arrow[r, "b_k"] \arrow[d, "f_{B,k}"'] & C_k \arrow[r, "c_k"] \arrow[d, "f_{C,k}"] & A_{k-1} \arrow[r, "a_{k-1}"] \arrow[d, "f_{A,k-1}"] & \cdots \\
\cdots \arrow[r, "c'_{k+1}"'] & A'_k \arrow[r, "a'_k"'] & B'_k \arrow[r, "b'_k"'] & C'_k \arrow[r, "c'_k"'] & A'_{k-1} \arrow[r, "a'_{k-1}"'] & \cdots
\end{tikzcd}$$

be a ($\mathbb{Z}$-indexed) commutative ladder in ${}_R\mathsf{Mod}$ with exact rows. Moreover, for every $k \in \mathbb{Z}$, let $f_{C,k}\colon C_k \longrightarrow C'_k$ be an isomorphism and let

$$\Delta_k := c_k \circ f_{C,k}^{-1} \circ b'_k \colon B'_k \longrightarrow A_{k-1}.$$

Show (via a diagram chase) that then the sequence

$$\cdots \xrightarrow{\Delta_{k+1}} A_k \xrightarrow{(f_{A,k}, -a_k)} A'_k \oplus B_k \xrightarrow{a'_k \oplus f_{B,k}} B'_k \xrightarrow{\Delta_k} A_{k-1} \longrightarrow \cdots$$

in ${}_R\mathsf{Mod}$ is exact.

Bonus problem (Abelian categories)**.**

1. Look up the term *Abelian category* in the literature.
2. Give an example for an additive category that is *not* Abelian.
3. How can exact sequences be defined in Abelian categories?
4. What does the Freyd–Mitchell embedding theorem say?

Submission before January 18, 2022, 8:30, via GRIPS