Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold/J. Witzig Sheet 12, January 18, 2022

Exercise 1 (small singular cycles)**.** Let X be a topological space, let $k \in \mathbb{N}_{>0}$. Which of the following statements are in this situation always true? Justify your answer with a suitable proof or counterexample.

1. If $\sigma \in \operatorname{map}(\Delta^k, X) \subset C_k(X)$ is a singular cycle of X, then k is odd.
2. If $\sigma, \tau \in \operatorname{map}(\Delta^k, X)$ and $\sigma + \tau$ is a singular cycle of X, then k is odd.

Exercise 2 (algebraic Euler characteristic)**.** Let R be a ring with unit that admits a nice notion rk_R of rank for finitely generated R-modules (e.g., fields, principal ideal rings, ...). A chain complex $C \in \operatorname{Ob}({}_R\mathsf{Ch})$ is *finite* if for every $k \in \mathbb{Z}$ the R-module C_k is finitely generated and $\{k \in \mathbb{Z} \mid C_k \not\cong_R 0\}$ is finite. The *Euler characteristic* of a finite chain complex $C \in \operatorname{Ob}({}_R\mathsf{Ch})$ is defined by

$$\chi(C) := \sum_{k \in \mathbb{Z}} (-1)^k \cdot \operatorname{rk}_R C_k.$$

Show that $\chi(C) = \sum_{k \in \mathbb{Z}} (-1)^k \cdot \operatorname{rk}_R \bigl(H_k(C)\bigr)$ and explain which properties of rk_R you used in your arguments.

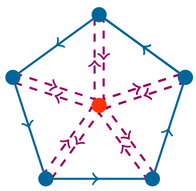
Exercise 3 (the ℓ^1-semi-norm on singular homology)**.** Let X be a topological space and let $k \in \mathbb{N}$. Let $|\cdot|_1$ be the ℓ^1-norm on $C_k(X;\mathbb{R})$ with respect to the $\mathbb{R}$-basis of $C_k(X;\mathbb{R})$ that consists of all singular k-simplices of X. We then define the *ℓ^1-semi-norm* $\|\cdot\|_1 \colon H_k(X;\mathbb{R}) \longrightarrow \mathbb{R}_{\geq 0}$ by

$$\|\alpha\|_1 := \inf\bigl\{|c|_1 \bigm| c \in C_k(X;\mathbb{R}),\ \partial_k c = 0,\ [c] = \alpha \in H_k(X;\mathbb{R})\bigr\}$$

for all $\alpha \in H_k(X;\mathbb{R})$.

1. Show that $\|\cdot\|_1$ is a semi-norm on $H_k(X;\mathbb{R})$.
2. Let $f \colon X \longrightarrow Y$ be a continuous map. Show that $\bigl\|H_k(f;\mathbb{R})(\alpha)\bigr\|_1 \leq \|\alpha\|_1$ holds for all $\alpha \in H_k(X;\mathbb{R})$.

Exercise 4 (singular homology in degree 1)**.** Let X be a path-connected, non-empty topological space. Let $\alpha \in H_1(X;\mathbb{Z})$. Show that there exists a continuous map $f \colon S^1 \longrightarrow X$ with $\alpha \in \operatorname{im} H_1(f;\mathbb{Z})$. Illustrate!



Bonus problem (realisation of homology groups)**.** Let $k \in \mathbb{N}_{>0}$. Construct a functor

$$R_k \colon {}_{\mathbb{Z}}\mathsf{Mod}^{\mathsf{fin}} \longrightarrow \mathsf{Top}_{\mathsf{h}}$$

with $h_k \circ R_k \cong \operatorname{Id}_{{}_{\mathbb{Z}}\mathsf{Mod}^{\mathsf{fin}}}$ and $h_\ell \circ R_k \cong 0$ for all $\ell \in \mathbb{N}_{>0} \setminus \{k\}$. Here, ${}_{\mathbb{Z}}\mathsf{Mod}^{\mathsf{fin}}$ denotes the category of all finitely generated $\mathbb{Z}$-modules.
Hints. Use spheres and mapping cones!

Submission before January 25, 2022, 8:30, via GRIPS