Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold/J. Witzig Sheet 13, January 25, 2022

Exercise 1 (chain homotopy equivalence)**.** Let R be a principal ideal domain and let $C, D \in \operatorname{Ob}({}_R\mathsf{Ch})$. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If C is finite and $C \simeq_{{}_R\mathsf{Ch}} D$, then also D is finite.
2. If C and D are finite with $C \simeq_{{}_R\mathsf{Ch}} D$, then $\chi(C) = \chi(D)$.

Hints. Finite chain complexes and the algebraic Euler characteristic were introduced in Exercise 2 on Sheet 12.

Exercise 2 (diameter of affine simplices)**.** Let $k \in \mathbb{N}$ and let $\sigma \colon \Delta^k \longrightarrow \mathbb{R}^\infty$ be an affine linear simplex. Show that every summand τ in the definition of the barycentric subdivision $B_k(\sigma)$ satisfies

$$\operatorname{diam}\bigl(\tau(\Delta^k)\bigr) \leq \frac{k}{k+1} \cdot \operatorname{diam}\bigl(\sigma(\Delta^k)\bigr).$$

Hints. For $A \subset \mathbb{R}^\infty$, we write $\operatorname{diam} A := \sup_{x,y \in A} \|x - y\|_2$. How can the diameter of affine linear simplices be expressed in terms of the vertices?

Exercise 3 (concrete cycles)**.** Give an example of a singular cycle in $C_2(S^2;\mathbb{Z})$ that represents a non-trivial class in $H_2(S^2;\mathbb{Z})$ and prove that this cycle indeed has this property. Illustrate!

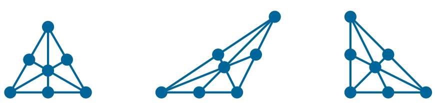
Exercise 4 (compatible homotopies)**.** Let X be a topological space, let $(S_k \subset \operatorname{map}(\Delta^k, X))_{k \in \mathbb{N}}$ be a family of simplices, and let $(h_\sigma)_{k \in \mathbb{N}, \sigma \in \operatorname{map}(\Delta^k, X)}$ be a family of homotopies with the following properties:

1. For each $k \in \mathbb{N}$ and each $\sigma \in \operatorname{map}(\Delta^k, X)$, the map $h_\sigma \colon \Delta^k \times [0,1] \longrightarrow X$ is a homotopy from σ to an element of S_k.
2. For all $k \in \mathbb{N}$, all $\sigma \in \operatorname{map}(\Delta^k, X)$, and all $j \in \{0, \dots, k\}$, we have
$$h_{\sigma \circ i_{k,j}} = h_\sigma \circ (i_{k,j} \times \mathrm{id}_{[0,1]}).$$
3. For all $k \in \mathbb{N}$ and all $\sigma \in S_k$, the homotopy h_σ satisfies
$$\forall_{x \in \Delta^k} \quad \forall_{t \in [0,1]} \quad h_\sigma(x,t) = \sigma(x).$$

Let $C^S(X) \subset C(X)$ be the subcomplex of the singular chain complex generated in each degree $k \in \mathbb{N}$ by S_k instead of $\operatorname{map}(\Delta^k, X)$.

Show that the inclusion $C^S(X) \longrightarrow C(X)$ is a chain homotopy equivalence in ${}_\mathbb{Z}\mathsf{Ch}$ (and thus induces an isomorphism in homology).

Bonus problem (barycentric subdivision)**.** Write a LaTeX-macro that draws the barycentric subdivision of affine 2-simplices (specified by their vertices):



Submission before February 1, 2022, 8:30, via GRIPS