Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold/J. Witzig Sheet 3, November 2, 2021

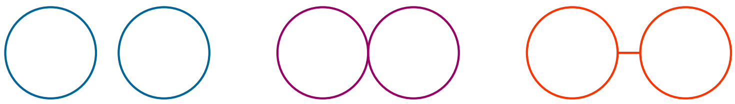
Exercise 1 (point-removal trick for homotopy equivalence?)**.** Let X and Y be topological spaces and let $x \in X$, $y \in Y$. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If $X \simeq Y$, then $X \setminus \{x\} \simeq Y \setminus \{y\}$.
2. If $X \setminus \{x\} \simeq Y \setminus \{y\}$, then $X \simeq Y$.

Exercise 2 (homotopy equivalence and path-connectedness)**.**

1. Show that a space X is path-connected if and only if all continuous maps $[0,1] \longrightarrow X$ are homotopic to each other.
2. Conclude that path-connectedness is preserved under homotopy equivalences.

Exercise 3 (OOOOOO)**.** We consider the folowing three subspaces of $\mathbb{R}^2$. State and prove the classification of these spaces up to homotopy equivalence.



Exercise 4 (mapping cones/Abbildungskegel)**.** Let X and Y be topological spaces and let $f \in \operatorname{map}(X,Y)$. The *mapping cone of f* is defined by the pushout

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & Y \\ {\scriptstyle i}\downarrow & & \downarrow \\ \operatorname{Cone}(X) & \longrightarrow & \operatorname{Cone}(f) \end{array}$$

where $\operatorname{Cone}(X) := (X \times [0,1])/(X \times \{0\})$ denotes the *cone over* X and $i \colon X \longrightarrow \operatorname{Cone}(X)$, $x \longmapsto [x,1]$ is the canonical map to the base of the cone. Prove the following statements (and illustrate your arguments graphically):

1. The cone $\operatorname{Cone}(X)$ is contractible if X is non-empty.
2. If $f \colon X \longrightarrow Y$ is a homotopy equivalence, then the mapping cone $\operatorname{Cone}(f)$ is contractible.

Bonus problem (four-point-circle)**.** Let $X := \{1,2,3,4\}$. We equip X with the topology generated by $\big\{\{1\},\{3\},\{1,2,3\},\{1,3,4\}\big\}$. In this exercise, you may use that S^1 is *not* contractible (which we will prove later in this course).

1. Show that there exists a surjective continuous map $S^1 \longrightarrow X$.
2. Show that S^1 and X are *not* homotopy equivalent.
3. *Bonus bonus problem.* Show that X is *not* contractible.

Submission before November 9, 2021, 8:30, via GRIPS