Algebraic Topology: Exercises

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Exercise 1 (spheres and π_n)**.** Let $n \in \mathbb{N}$. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If $\pi_n(S^n, e_1)$ consists of a single element, then for every pointed topological space (X, x_0) also $\pi_n(X, x_0)$ consists of a single element.
2. If $\pi_n(S^n, e_1)$ is infinite, then for every pointed space (X, x_0) also $\pi_n(S^n \times X, (e_1, x_0))$ is infinite.

Exercise 2 (product in $\mathsf{Top_h}$)**.** Let $(X_i)_{i\in I}$ be a family of topological spaces. Show that $\prod_{i\in I} X_i$ together with the homotopy classes of the canonical projections onto the factors satisfies the universal property of the product of $(X_i)_{i\in I}$ in the homotopy category $\mathsf{Top_h}$.

Exercise 3 (invariance of the boundary)**.** In this exercise, you may assume that the theorem on existence of "interesting" homotopy invariant functors holds.

Let $n \in \mathbb{N}_{>0}$ and let $H^n := \{x \in \mathbb{R}^n \mid x_n \geq 0\}$ be the upper half-space (with respect to the subspace topology of $\mathbb{R}^n$). Prove *invariance of the boundary*, i.e., show that there is no open neighbourhood of 0 in H^n that is homeomorphic to the open unit ball $(D^n)^\circ$ in $\mathbb{R}^n$.

Hints. As first step, prove the following: If $U \subset H^n$ is an open neighbourhood of 0 in H^n, then $U \setminus \{0\} \simeq U$.

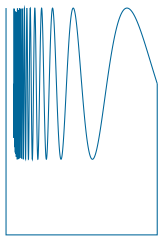
Exercise 4 (the Möbius strip is not boring)**.** In this exercise, you may assume that the theorem on existence of "interesting" homotopy invariant functors holds.

1. Show that the Möbius strip is not homeomorphic to $S^1 \times [0, 1]$.

   Hints. Use Exercise 3.

2. Is the Möbius strip homotopy equivalent to $S^1 \times [0, 1]$? Justify!

Bonus problem (Warsaw circle/Warschauer Kreis)**.** The topological space



$$W := \bigl\{(x, \sin(2 \cdot \pi/x)) \bigm| x \in (0, 1]\bigr\} \\ \cup \bigl(\{1\} \times [-2, 0]\bigr) \cup \bigl([0, 1] \times \{-2\}\bigr) \cup \bigl(\{0\} \times [-2, 1]\bigr)$$

(endowed with the subspace topology of $\mathbb{R}^2$) is called *Warsaw circle*. Prove that for every basepoint $w_0 \in W$ the fundamental group $\pi_1(W, w_0)$ is trivial.

Hints. Show first that no loop in W can cross the "gap."

Bonus problem (Nash equilibria)**.** Look up the notion of Nash equilibria in Game Theory. Prove the existence of Nash equilibria using the Brouwer fixed point theorem.

Submission before November 16, 2021, 8:30, via GRIPS