Algebraic Topology: Exercises

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Hints. When writing up your solutions, for each problem, first briefly explain the underlying idea and then carry out the arguments in detail.

Hints. You may use that the following map is a group isomorphism:

$$\begin{aligned} \mathbb{Z} &\longrightarrow \pi_1(S^1, e_1) \\ d &\longmapsto \bigl[[t] \mapsto [d \cdot t \mod 1]\bigr]_* \end{aligned}$$

Exercise 1 (injectivity/surjectivity and π_1)**.** Let $f\colon (X, x_0) \longrightarrow (Y, y_0)$ be a pointed continuous map between pointed spaces. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If f is injective, then $\pi_1(f)\colon \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$ is injective.
2. If f is surjective, then $\pi_1(f)\colon \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$ is surjective.

Exercise 2 (π_0 and path-connected components)**.** Look up the definition of path-connected components of topological spaces. Let (X, x_0) be a pointed space and let $\mathrm{PC}(X)$ be the set of path-connected components of X. Prove that the following map is a well-defined bijection:

$$\begin{aligned} \pi_0(X, x_0) &\longrightarrow \mathrm{PC}(X) \\ [\gamma]_* &\longmapsto [\gamma(-1)] \end{aligned}$$

Exercise 3 (π_1 and contractibility)**.** Let X be a topological space.

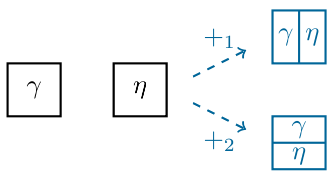
1. Let $\gamma\colon S^1 \longrightarrow X$ be a null-homotopic map and let $x_0 := \gamma(1)$. Show that then $[\gamma]_*$ is trivial in $\pi_1(X, x_0)$. Illustrate your argument in a suitable way!
2. Conclude: If X is contractible (but not necessarily pointedly contractible!) and $x_0 \in X$, then $\pi_1(X, x_0)$ is the trivial group.

Exercise 4 (fundamental theorem of algebra)**.** Use $\pi_1(S^1, e_1)$ to prove that every non-constant polynomial in $\mathbb{C}[X]$ has at least one root in $\mathbb{C}$.
Hints. Show that a non-constant polynomial $p \in \mathbb{C}[X]$ without roots in $\mathbb{C}$ would yield a map $S^1 \longrightarrow S^1$ that is both null-homotopic and homotopic to the map $S^1 \longrightarrow S^1$, $[t] \mapsto [\deg p \cdot t \mod 1]$, …

Bonus problem (group structure on π_n)**.** Let $n \in \mathbb{N}_{\geq 2}$ and let the composition maps $+_1, \dots, +_n\colon \pi_n(X, x_0) \times \pi_n(X, x_0) \longrightarrow \pi_n(X, x_0)$ be defined as in Outlook 2.1.6.

1. Prove that $+_j = +_1$ for all $j \in \{1, \dots, n\}$.
2. Prove that $\pi_n(X, x_0)$ is an Abelian group with respect to $+_1$.

Hints. Eckmann and Hilton might help!



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