Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold/J. Witzig Sheet 7, November 30, 2021

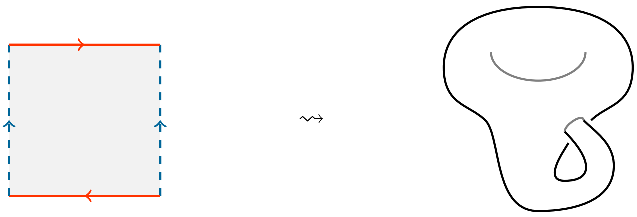
Hints. You may use that the following map is a group isomorphism:

$$\begin{aligned} \mathbb{Z} &\longrightarrow \pi_1(S^1, e_1) \\ d &\longmapsto \bigl[[t] \mapsto [d \cdot t \mod 1]\bigr]_* \end{aligned}$$

Exercise 1 (coverings)**.** Let X, Y, Z be topological spaces and let $p \colon X \longrightarrow Y$, $q \colon Y \longrightarrow Z$ be continuous maps. Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. If $q \circ p$ is a covering map, then also q is a covering map.
2. If $q \circ p$ is a covering map, then also p is a covering map.

Exercise 2 (Klein bottle/Kleinsche Flasche)**.** Let K be the *Klein bottle*, i.e., the quotient space $K = ([0,1] \times [0,1])/{\sim}$ defined by the following glueing relation:



1. Show that the fundamental group of K (at the basepoint $([0],[0])$) is non-Abelian.
2. Show that there exists a 2-sheeted covering $S^1 \times S^1 \longrightarrow K$. Draw it!

Exercise 3 (pretzel coverings)**.** Let $(B, b) := (S^1, e_1) \vee (S^1, e_1)$. Solve one of the following:

1. Construct (and draw) two connected 2-sheeted coverings of (B, b) that are *not* isomorphic in $\mathsf{Cov}_{(B,b)}$ (and prove that they are not isomorphic).
2. Construct (and draw) a connected 3-sheeted covering of (B, b) whose deck transformation group does *not* act transitively on the fibres (and prove this fact).

Exercise 4 (coverings of wild spaces)**.** Solve one of the following

1. Show that the Warsaw circle (Bonus problem of Sheet 4) admits a non-trivial covering. Draw it!

   Hints. Warsaw helix!

2. Show that the Hawaiian earring (Bonus problem of Sheet 6) admits *no* covering with simply connected (non-empty) total space.

Please turn over

Bonus problem (one-dimensional complexes I). A *one-dimensional complex* is a pair (X, X_0), consisting of a topological space X and a discrete subspace X_0 with the following property: There exists a set I and a pushout (in Top) of the form

$$\begin{tikzcd} \coprod_I S^0 \arrow[r] \arrow[d] & X_0 \arrow[d] \\ \coprod_I D^1 \arrow[r] & X \end{tikzcd}$$

where the left vertical arrow is the canonical inclusion and the right vertical arrow is the inclusion of X_0 into X. I.e., one-dimensional complexes can be obtained by glueing intervals at their end-points.

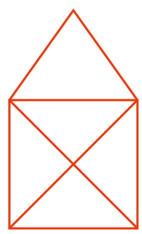
Such a one-dimensional complex is *finite* if both X_0 and I are finite.

1. Let (X, X_0) be a one-dimensional complex and let $x_0 \in X_0$. What does the spread of infectious deseases have to do with $\pi_0(X, x_0)$?

   Hints. Take X_0 as a set of people, take $\coprod_I D^1$ and the glueing maps as ...

2. Find an algorithm that given a finite one-dimensional complex (X, X_0) determines whether X is path-connected or not. In particular, explain how you represent (X, X_0) by finite data, why your algorithm terminates and why your algorithm produces the correct result.

Nikolaus problem (Haus des Nikolaus, all covered). The traditional *Haus des Nikolaus* is the following subspace of $\mathbb{R}^2$:



The new edition of this house will be built by the Blorx Building Trust (which won this contract in a transparent, corruption-free procedure).

Given $n \in \mathbb{N}$, the Blorx Building Trust will construct a path-connected n-sheeted covering space of the Haus des Nikolaus. Write a LaTeX macro \nikolaus with one argument such that \nikolaus$\{n\}$ draws a beautiful path-connected n-sheeted covering of the Haus des Nikolaus. Execute \nikolaus$\{8\}$.

Hints. If you have never seen the Haus des Nikolaus:
http://www.mathematische-basteleien.de/house.html

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