Algebraic Topology: Exercises

Prof. Dr. C. Löh/M. Uschold/J. Witzig — Sheet 8, December 3, 2018

Hints. You may use that the following map is a group isomorphism:

$$\begin{aligned} \mathbb{Z} &\longrightarrow \pi_1(S^1, e_1) \\ d &\longmapsto \bigl[[t] \mapsto [d \cdot t \mod 1]\bigr]_* \end{aligned}$$

Exercise 1 (coverings)**.** Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. There exists a covering map $\mathbb{R}P^2 \longrightarrow S^1 \times S^1$.
2. There exists a covering map $\mathbb{C} \setminus \{0, 1\} \longrightarrow \mathbb{C} \setminus \{0\}$.

Exercise 2 (counting sheets)**.** Let $X := S^{2020} \times \mathbb{R}P^{2021} \times \mathbb{R}P^{2022}$. Use the π_1-action on the fibers of covering maps to show that there are no path-connected coverings of X that have

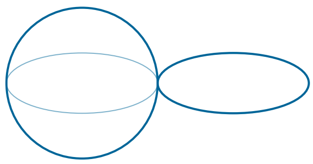
1. infinitely many sheets; or
2. exactly three sheets.

Exercise 3 (The Borsuk-Ulam theorem in dimension 2)**.** A map $f\colon S^2 \longrightarrow S^1$ is *antipodal* if $f(-x) = -f(x)$ holds for all $x \in S^2$.

1. Show that there is no continuous antipodal map $S^2 \longrightarrow S^1$.

   Hints. Try to use $\mathbb{R}P^2$ and consider a path in S^2 from e_1 to $-e_1$.
2. Prove the *Borsuk-Ulam theorem in dimension* 2: If $f\colon S^2 \longrightarrow \mathbb{R}^2$ is continuous, then there exists $x \in S^2$ with $f(x) = f(-x)$.

Exercise 4 (large homotopy groups)**.** Let $(X, x_0) := (S^2, e_1) \vee (S^1, e_1)$.



1. Give a simple geometric description of the universal covering of X and prove that this indeed is a universal covering of X.
2. Let $k \in \mathbb{N}_{\geq 2}$ with the property that $\pi_k(S^2, e_1^2)$ is non-trivial. Prove that then $\pi_k(X, x_0)$ is *not* finitely generated (as Abelian group).

Bonus problem (one-dimensional complexes II)**.** We consider one-dimensional complexes as in the Bonus problem on Sheet 7.

1. Let (X, X_0) be a one-dimensional complex and let $x_0 \in X_0$. Show that $\pi_1(X, x_0)$ is a free group.

   Hints. You may restrict to the finite case.
2. Let (X, X_0) be a one-dimensional complex and let $p\colon Y \longrightarrow X$ be a covering map. Show that then also $(Y, p^{-1}(X_0))$ is a one-dimensional complex.

Submission before December 10, 2018, 8:30, via GRIPS