Algebraic Topology: Exercises

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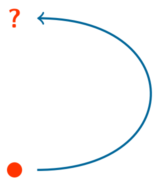
Exercise 1 (regular coverings?). Which of the following statements are true? Justify your answer with a suitable proof or counterexample.

1. All path-connected coverings of $\mathbb{R}P^{2021} \times \mathbb{R}P^{2022}$ are regular.
2. All path-connected coverings of $(\mathbb{R}P^{2021}, [e_1]) \vee (\mathbb{R}P^{2022}, [e_1])$ are regular.

Exercise 2 (no finite coverings). Show that there exists a path-connected, locally path-connected, semi-locally simply connected, non-empty topological space that is *not* simply connected and does *not* admit a finite-sheeted non-trivial covering.
Hints. Don't be irrational!

Exercise 3 (residually finite groups/residuell endliche Gruppen). A group G is *residually finite* if the following holds: For every $g \in G \setminus \{e\}$ there exists a finite group F and a group homomorphism $\varphi\colon G \longrightarrow F$ with $\varphi(g) \neq e$.

1. Let (X, x_0) be a pointed space that admits a universal covering. Give an equivalent characterisation of residual finiteness of $\pi_1(X, x_0)$ in terms of coverings of (X, x_0) and (lifts of) paths.
2. Illustrate these concepts for the circle $(S^1, 1)$.



Exercise 4 (the Heisenberg manifold). We consider the *Heisenberg group*

$$H := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{Z} \right\} \subset H_{\mathbb{R}} := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, z \in \mathbb{R} \right\} \subset \mathrm{SL}_3(\mathbb{R})$$

and the *Heisenberg manifold* $M := H \setminus H_{\mathbb{R}}$, where H acts on $H_{\mathbb{R}}$ by matrix multiplication. Solve two of the following questions and give detailed proofs for your answers:

1. What is the fundamental group of M ?
2. Is there a covering of the 3-torus by M ?
3. Is there a covering of M by the 3-torus ?
4. Does M admit a path-connected regular covering whose deck transformation group is isomorphic to $\mathbb{Z}^2$?

Please turn over

Bonus problem (classifying spaces of torsion-free groups). Let G be a group and let $\mathrm{P}_{\mathrm{fin}}(G)$ denote the set of all finite subsets of G. For $S \in \mathrm{P}_{\mathrm{fin}}(G)$ we write

$$\Delta_G(S) := \Bigl\{ f \colon G \longrightarrow [0,1] \Bigm| f|_{G \setminus S} = 0 \text{ and } \sum_{s \in S} f(s) = 1 \Bigr\}.$$

We write $\Delta(G) := \bigcup_{S \in \mathrm{P}_{\mathrm{fin}}(G)} \Delta_G(S)$ with the corresponding colimit topology (i.e., a subset of $\Delta(G)$ is closed if and only if the intersection with $\Delta_G(S)$ is closed for all $S \in \mathrm{P}_{\mathrm{fin}}(G)$, where $\Delta_G(S)$ carries the Euclidean topology). Furthermore, we consider the continuous left action

$$\begin{aligned} G \times \Delta(G) &\longrightarrow \Delta(G) \\ (g, f) &\longmapsto \bigl(h \mapsto f(g^{-1} \cdot h)\bigr) \end{aligned}$$

of G on $\Delta(G)$. Solve two of the following:

1. Show that $\Delta(G)$ is pointedly contractible with respect to every basepoint.
2. Determine the fundamental group of the quotient $G \setminus \Delta(G)$ in the case that the group G is torsion-free.
3. Extend this construction to an interesting functor $\mathsf{Group} \longrightarrow \mathsf{Top}$.

Submission before December 21, 2021, 8:30, via GRIPS