Algebraic Topology: Études

Prof. Dr. C. Löh/M. Uschold/J. Witzig Sheet 3, November 5, 2021

Exercise 1 (homotopy). Let X be a topological space and let $f \in \operatorname{map}(X, X)$. Prove or disprove:

1. If $f \simeq \mathrm{id}_X$, then f is bijective.
2. If $f \simeq \mathrm{id}_X$, then $f \circ f \simeq f$.
3. If $f \circ f \simeq f$, then $f \simeq \mathrm{id}_X$.
4. If $f^{2021} \simeq \mathrm{id}_X$, then f is a homotopy equivalence.

Exercise 2 (homotopy invariant functors). Let $F \colon \mathsf{Top} \longrightarrow \mathsf{Ab}$ be a homotopy invariant functor. Prove or disprove:

1. $F(D^{2021}) \cong_{\mathsf{Ab}} F(\mathbb{R}^{2022})$.
2. $F(S^{2021}) \not\cong_{\mathsf{Ab}} \mathbb{Z}^{2021}$.
3. $F(\mathbb{R}P^{2021}) \cong_{\mathsf{Ab}} F(\mathbb{R} \times \mathbb{R}P^{2021})$.
4. If X is contractible, then $F(X) \cong_{\mathsf{Ab}} \{0\}$.
5. If $F(\mathbb{R}P^{2021}) \cong_{\mathsf{Ab}} \mathbb{Z}$, then $F(S^{2021}) \not\cong_{\mathsf{Ab}} \mathbb{Z}$.

Hints. Recall that all of these problems are easy!

Exercise 3 (classification problem). In this exercise, you may assume that the theorem on existence of "interesting" homotopy invariant functors holds. Classify the following spaces up to homeomorphism/homotopy equivalence.

1. $\mathbb{R}^{2021}$
2. $\mathbb{R}^{2022}$
3. S^{2021}
4. D^{2022}
5. $S^0 \times S^{2021}$
6. $S^0 \times S^{2022}$

Exercise 4 (summary). Write a summary of Chapter 1.3 (Homotopy and Homotopy Invariance), keeping the following questions in mind:

1. What is homotopy/homotopy equivalence?
2. What are basic examples?
3. What is homotopy invariance?
4. How can homotopy invariance be used?

No submission!