Algebraic Topology: Études

Prof. Dr. C. Löh/M. Uschold/J. Witzig Sheet 9, December 17, 2021

Exercise 1 (exact sequences). Which of the following sequences of $\mathbb{Z}$-modules are exact?

1. $\mathbb{Z} \xrightarrow{2021} \mathbb{Z} \xrightarrow{2021} \mathbb{Z}$
2. $\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$
3. $\mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2021} \mathbb{Z}$
4. $\mathbb{Z} \xrightarrow{2021} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$

Exercise 2 (long exact sequences). Let $\bigl((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\bigr)$ be an ordinary homology theory on Top^2 and let (X, A) be a pair of spaces.

1. Write down the long exact sequence of this pair with respect to the homology theory $\bigl((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}}\bigr)$. What can you conclude from this sequence if X is contractible? What if A is contractible?
2. Apply this to $(\{0\}, \{0\})$.
3. Apply this to $(\mathbb{R}^2, \mathbb{R}^2 \setminus S^1)$.
4. Apply this to $(S^1, \{e_1\})$.

Exercise 3 (excision). Which of the following pairs of spaces are related by excision (as in the excision axiom)?

1. $(\mathbb{R}^{2021}, \{0\})$ and $(\mathbb{R}^{2021} \setminus \{0\}, \emptyset)$
2. $(\mathbb{R}^2, S^1)$ and $(\mathbb{R}^2 \setminus \{0\}, \emptyset)$
3. $(\mathbb{R}^2, \mathbb{R}^2 \setminus \{0\})$ and $([0,1] \times [0,1], ([0,1] \times [0,1]) \setminus \{0\})$
4. $(\mathbb{R}^2, \mathbb{R}^2 \setminus \{2 \cdot e_1\})$ and $(\mathbb{R}^2 \setminus D^2, \mathbb{R}^2 \setminus (D^2 \cup \{2 \cdot e_1\}))$
5. $(\mathbb{R}^2, \mathbb{R}^2 \setminus \{e_1\})$ and $(\mathbb{R}^2 \setminus D^2, \mathbb{R}^2 \setminus D^2)$
6. $(\mathbb{R}^2, \mathbb{R}^2 \setminus \{0, 2 \cdot e_1\})$ and $(D^2, D^2 \setminus \{0\})$

Exercise 4 (summary). Write a summary of Chapter 2.3 (Covering Theory) and Chapter 2.4 (Applications), keeping the following questions in mind:

1. What are important examples of (non-trivial) coverings?
2. Which lifting properties do coverings have? Why?
3. Why are coverings compatible with homotopy groups?
4. How can coverings be classified?
5. How can covering theory be used to compute fundamental groups?
6. Which applications do fundamental groups and covering theory have?

No submission!