Recap: Classification Theorem. 14.12.2021

Let X be a path-conn., loc. path-conn., semi-loc. simply conn. space, let $x_0 \in X$, let $p : \tilde{X} \to X$ be "the" universal covering of X, and let $\tilde{x}_0 \in p^{-1}(x_0)$.

1. Then the following are mutually "inverse" nat. equivalences:

$$\mathrm{Cov}^0_{(X,x_0)} \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathrm{Subgroup}_{\pi_1(X,x_0)}$$

obj: $\big(q : (Y,y_0) \to (X,x_0)\big) \longmapsto \pi_1(q)\big(\pi_1(Y,y_0)\big)$

$$\left(\left(H\backslash\tilde{X}, H\tilde{x}_0\right) \xrightarrow{q_H} (X,x_0) \atop Hx \longmapsto p(x) \right) \longleftarrow H \qquad \text{obj}$$

Proof 1. q_H is a covering: let $G := \pi_1(X,x_0)$.

If $H < G$, then $q_H : H\backslash\tilde{X} \to X$ is a covering:
$$Hx \longmapsto p(x)$$

- q_H is well-def. (G acts by deck trafos on $\tilde{X}$)
- covering property: By construction:

$$\underbrace{p}_{\text{covering}} = q_H \circ \underbrace{p_H}_{\text{covering (prop. disc. action)}}$$

can. proj. $\tilde{X} \longrightarrow H\backslash\tilde{X}$

$\leadsto$ q_H is a covering (check!)

- $q_H(H\tilde{x}_0) = x_0$

Hence: $q_H \in \mathrm{Ob}\left(\mathrm{Cov}^0_{(X,x_0)}\right)$

① φ is a functor (π_1 is a functor)
② ψ is a functor (calculation)

③ $\varphi \circ \psi = \mathrm{Id}_{\mathrm{Subgrp}_G}$: let $H < G$. Then

$$\varphi \circ \psi(H) = \underbrace{\pi_1(q_H)\left(\pi_1\left(H\backslash\widetilde{X}, H\widetilde{x}_0\right)\right)}$$

$\boxed{*}$ stab. of $H\widetilde{x}_0$ w.r.t. to the action of $\pi_1(X, x_0)$ on $q_H^{-1}(x_0)$ (Cor. 2.3.15)

Relate this to the $\pi_1(X, x_0)$-action on $p^{-1}(x_0)$:

let $\gamma \in \mathrm{map}_*((S^1, e_1), (X, x_0))$, let $\widetilde{\gamma} : [0,1] \to \widetilde{X}$ be the p-lift of $\bar{\gamma} : t \mapsto \gamma([t])$ starting at $\widetilde{x}_0$.

Then: $p_H \circ \widetilde{\gamma} : [0,1] \to H\backslash\widetilde{X}$ is the q_H-lift of $\bar{\gamma}$ to $H\backslash\widetilde{X}$ starting at $H\widetilde{x}_0$.

Hence: $[\gamma]_* \in \boxed{*}$ if and only if

$$p_H \circ \widetilde{\gamma}(0) = p_H \circ \widetilde{\gamma}(1).$$

This is equiv to $[\widetilde{\gamma}]_* \in H$.

Thus: $\varphi \circ \psi(H) = H$.

- on morphisms: $\varphi \circ \psi$ maps incls of subgroups to the same incl.

(4) $\psi\circ\varphi \cong \mathrm{Id}_{\mathrm{Cov}^\circ(X,x_0)}$: let $q:(Y,y_0)\to(X,x_0)$ be in $\mathrm{Cov}^\circ(X,x_0)$; let $H := \varphi(q)$

$$= \pi_1(q)(\pi_1(Y,y_0)) < G$$

By def: $\psi\circ\varphi(q) = q_H$.

It suffices to show that there ex. exactly one iso in $\mathrm{Cov}^\circ_{(X,x_0)}$ between q and q_H. (+ calc.)

$$\begin{tikzcd}
(Y,y_0) \arrow[r, shift left, "f"] \arrow[d, "q"', "H"] & (\tilde{X}_H, H\tilde{x}_0) \arrow[l, shift left, "g"] \arrow[d, "q_H", "H"'] \\
(X,x_0) \arrow[r, equal] & (X,x_0)
\end{tikzcd}$$

by (3)

By the π_1-lifting crit.:

- there ex. a unique $f\in \mathrm{map}_*((Y,y_0),(\tilde{X}_H, H\tilde{x}_0))$ with $q_H\circ f = q$
- there ex. a unique $g\in \mathrm{map}_*((\tilde{X}_H, H\tilde{x}_0),(Y,y_0))$ with $q\circ g = q_H$.

Moreover, $f\circ g = \mathrm{id}_{\tilde{X}_H}$ and $g\circ f = \mathrm{id}_Y$ by the uniqueness of lifts.

$\rightsquigarrow$ $q \cong_{\mathrm{Cov}^\circ(X,x_0)} q_H$ (and the iso is unique).

$\rightsquigarrow$ $\psi\circ\varphi \cong \mathrm{Id}_{\mathrm{Cov}^\circ(X,x_0)}$

Classification Theorem

2. Let $(q:(Y,y_0)\to(X,x_0))\in \mathrm{Cov}^0_{(X,x_0)}$ and $H:=\varphi(q)$
$=\pi_1(q)(\pi_1(Y,y_0))$.

Then:

(a) q is $[\pi_1(X,x_0):H]$-sheeted

(b) $\mathrm{Deck}(q)\cong N_{\pi_1(X,x_0)}(H)/H$

(c) q is regular if and only if $H\trianglelefteq \pi_1(X,x_0)$

Proof. 2. In view 1: suffices to prove (a), (b), (c) for q_H instead of q.

(a) Follows from

$$|q_H^{-1}(x_0)| = |\{H\cdot g\cdot\tilde{x}\mid g\in G\}|$$
$$= |\{H\cdot g\mid g\in G\}| = [G:H].$$

$G\curvearrowright\tilde{X}$ freely

$\{g\in G\mid Hg=gH\}$

(b) We consider $f: N_G H\longrightarrow \mathrm{Deck}(q_H)$
$g\longmapsto f_g$,

- $f_g: H\backslash\tilde{X}\longrightarrow H\backslash\tilde{X}$
  $H\cdot x\longmapsto H\cdot g\cdot x = gHx$

  deck transfo action $G\curvearrowright\tilde{X}$

  well-def: because $g\in N_G H$

- f is hom ✓

- $\ker(f) = H$: By constr.: $H \subset \ker f$.

Conversely: let $g \in \ker f$. Then

$$H \cdot g \tilde{x}_0 = f_g(H\tilde{x}_0) = H\tilde{x}_0$$

$G \curvearrowright \tilde{X}$ free

$H \cdot g = H$, so $g \in H$.

$\rightsquigarrow \ker f = H$.

- $\operatorname{im} f = \operatorname{Deck}(q_H)$: Let $d \in \operatorname{Deck}(q_H)$.

Because $G \curvearrowright \tilde{X}$ is transitive on $p^{-1}(x_0)$, there ex. $g \in G$ with

$$d(H\tilde{x}_0) = H \cdot g \cdot \tilde{x}_0 .$$

Then $g \in N_G H$, because:

$$H \overset{!}{=} \pi_1(q_H)(\pi_1(H\backslash\tilde{X}, H\tilde{x}_0))$$

$q_H \circ d = q_H$

$$= \pi_1(q_H) \circ \pi_1(d)(\pi_1(H\backslash\tilde{X}, H\tilde{x}_0))$$

d is a homeo

$$= \pi_1(q_H)(\pi_1(H\backslash\tilde{X}, Hg\tilde{x}_0))$$

$$= \pi_1(q_{g^{-1}Hg})(\pi_1(g^{-1}Hg\backslash\tilde{X}, g^{-1}Hg\tilde{x}_0))$$

$$\overset{!}{=} g^{-1}Hg$$

$$\begin{tikzcd} \tilde{X} \arrow[rr, "\text{"}x \mapsto g^{-1}x\text{"}"] \arrow[dd, "q_H"'] & & \tilde{X} \arrow[dd, "q_{g^{-1}Hg}"] \\ H \backslash \tilde{X} \arrow[rr] & & g^{-1}Hg \backslash \tilde{X} \\ X \arrow[rr, equal] & & X \end{tikzcd}$$

Therefore: $g \in N_G H$.

$\rightsquigarrow f$ induces an iso $N_G H / H \cong \mathrm{Deck}(q_H)$.

ⓒ The proof of ⓑ shows: $\mathrm{Deck}(q_H)$ acts transitively on $q^{-1}(x_0)$ if and only if $\underbrace{N_G(H) = G}_{\Longleftrightarrow\ H \trianglelefteq G}$. $\square$

Example (simply connected spaces). Let X be a simply conn., $\underbrace{\text{loc. path-connected}}_{\text{important!}}$ space, let $x_0 \in X$.

applicable!

ⓗ 7.4

$\rightsquigarrow$ By the classif. thm: All (path-conn.) coverings of X are trivial!

Why? $\cdot\ \pi_1(X, x_0) \cong 1$ has no non-trivial subgroups

$\cdot$ "the" covering corresp. to $1 < \pi_1(X, x_0)$ is $\mathrm{id}: (X, x_0) \to (X, x_0)$, which is trivial.

Example. (circle). What is $\mathrm{Cov}^{o}_{(S^1,e_1)}$?

The classif. thm is applicable.

$\rightsquigarrow$ $\mathrm{Cov}^{o}_{(S^1,e_1)}$ is classified by subgroups $\underbrace{\pi_1(S^1,e_1)}_{\cong \mathbb{Z}}$

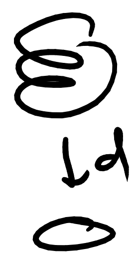
Each subgroup of $\mathbb{Z}$ is of the form $d\cdot\mathbb{Z}$ with $d \in \mathbb{N}$.

$0\cdot\mathbb{Z} < \mathbb{Z}$: The univ. cov. $(\mathbb{R},0) \to (S^1,e_1)$

$1\cdot\mathbb{Z} < \mathbb{Z}$: $\mathrm{id}_{S^1} : (S^1,e_1) \to (S^1,e_1)$

$d\cdot\mathbb{Z} < \mathbb{Z}$: $(S^1,e_1) \to (S^1,e_1)$

with $d > 1$ $\quad [t] \mapsto [d\cdot t \bmod 1]$



up to iso this is all of $\mathrm{Cov}^{o}_{(S^1,e_1)}$.

Example. (double coverings) (of "nice" spaces)

All double coverings (with path-conn. total space) are regular, because: all subgroups of index 2 are normal!

2-sheeted

Goal: understand 2-sheeted path-conn. coverings of $(S^1,e_1)\vee(S^1,e_1)$.
(classif. thm is applicable!)

$\pi_1(\dots) \cong F_2 \cong F(a,b)$

→ need to understand all index 2 subgroups of $F(a,b)$, which is the same as underst. all epis $F(a,b) \longrightarrow \mathbb{Z}/2$.

What are these epis? These are exactly the following:

$F(a,b) \to \mathbb{Z}/2$ free group!

not an epi: $a \mapsto [0]$, $b \mapsto [0]$

epi A: $a \mapsto [1]$, $b \mapsto [0]$

epi B: $a \mapsto [0]$, $b \mapsto [1]$

epi AB: $a \mapsto [1]$, $b \mapsto [1]$

Next time: figure out the cov's for ker A, ker B, ker AB.