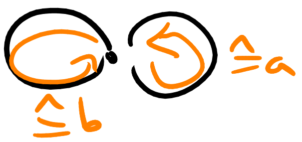
Recap: Classification theorem 17.12.2021

$$\mathrm{Cov}^0_{(X,x_0)} \longleftrightarrow \mathrm{Subgroups}\ \pi_1(X,x_0)$$
$$q \longmapsto \mathrm{im}\,\pi_1(q)$$

Example (ctd.). Double coverings of $(S^1,e_1)\vee(S^1,e_1) =: (X,x_0)$?

Classif. thm ⇝ need to understand all index 2 subgroups of $\pi_1(X,x_0) \cong F(a,b)$



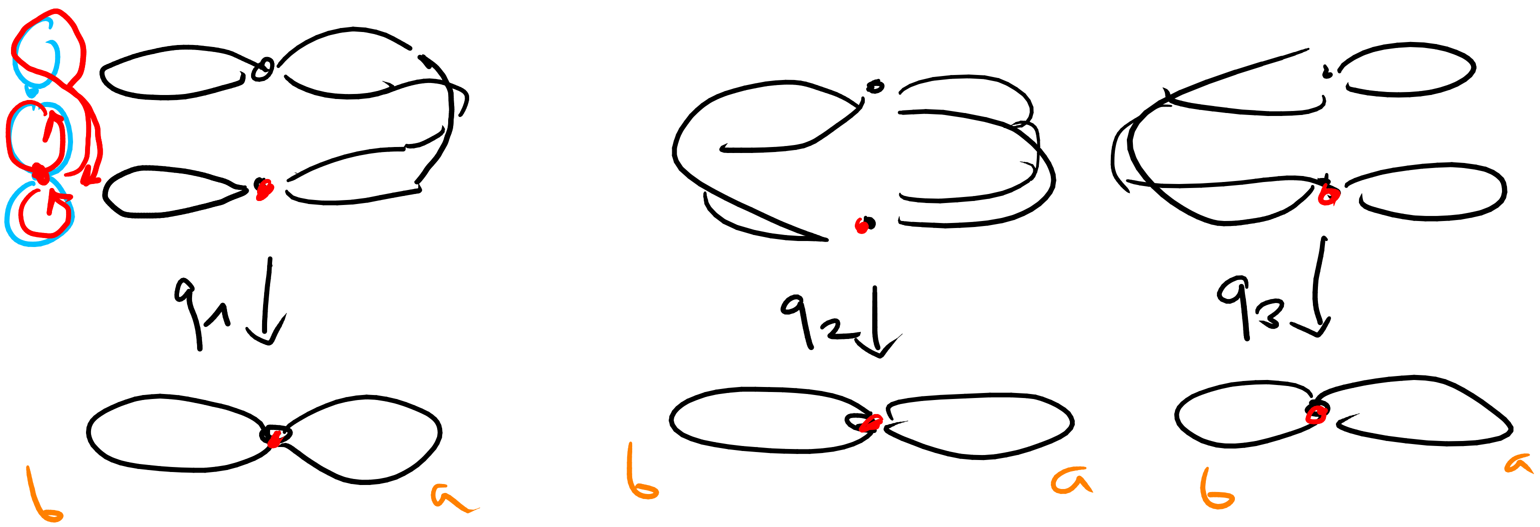
i.e.: need to understand the kernels of all epis $F(a,b) \to \mathbb{Z}/2$

exactly three:

$A: F(a,b) \to \mathbb{Z}/2$, $a \mapsto [1]$, $b \mapsto [0]$

$AB: F(a,b) \to \mathbb{Z}/2$, $a \mapsto [1]$, $b \mapsto [1]$

$B: F(a,b) \to \mathbb{Z}/2$, $a \mapsto [0]$, $b \mapsto [1]$



three "diff" (pw not iso in $\mathrm{Cov}^0_{(X,x_0)}$) double coverings of (X,x_0)

⇝ these are "all" double coverings of (X,x_0)

up to iso, ptd, path-conn.

Which double cov. corresponds to which epi?

$b \in \mathrm{im}\,\pi_1(q_1)$, $a \in \mathrm{im}\,\pi_1(q_3)$

$\rightsquigarrow$ only one possible matching:

$$q_1 \mathrel{\hat{=}} \ker A, \quad q_3 \mathrel{\hat{=}} \ker B, \quad q_2 \mathrel{\hat{=}} \ker AB$$

From this correspondence, as a bonus, we get:

$\ker A$ is gen by b, a^2, aba^{-1}

$\ker B$ " a, b^2, bab^{-1}

$\ker AB$ " a^2, b^2, ab.

$\boxed{2.3.6}$ APPLICATION: THE NIELSEN–SCHREIER THEOREM

Recall: let R be a field, $\mathbb{Z}$, PID, ... not true in gen!

Then: submodules of free R-modules are free R-modules

Theorem (Nielsen–Schreier). Subgroups of free groups are free (!)

$$\begin{tikzcd} S \arrow[r, "\text{map}"] \arrow[d] & G \\ F \arrow[ur, dashed, "\exists!\,\text{hom}"'] & \end{tikzcd}$$

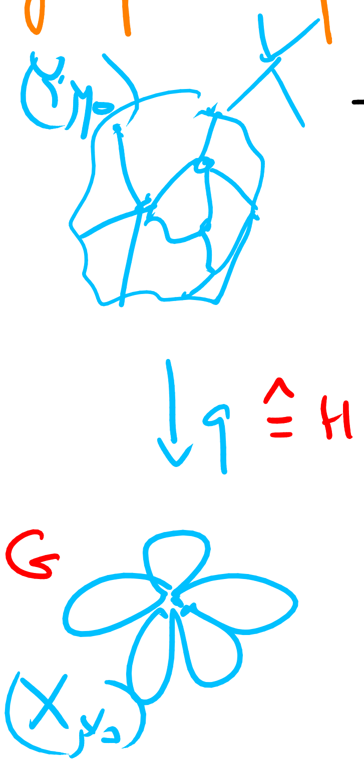
e.g.: $F(a,b)$ is free

· $\mathbb{Z}/2$ or $\mathbb{Z}^2$ are not free

Proof (via topology). idea: use covering theory!

Let G be a free group with free generating set S and let $H \subset G$ be a subgroup.
(need to show: H is free)

groups ~> top. We consider $(X, x_0) := \bigvee_S (S^1, e_1)$.



Then: $\pi_1(X, x_0) \cong F(S) \cong G$.

f $\cong$

By the classif. thm of coverings, there ex. a path-conn. covering

$$q : (Y, y_0) \to (X, x_0) \text{ with}$$

$$\pi_1(q)\big(\pi_1(Y, y_0)\big) = f^{-1}(H).$$

exploit topology: X has the structure of a "one-dim" complex (Ⓣ 7.B / 8.B)

Ⓣ 8.B
~> also Y has the structure of a one-dim complex.

Ⓣ 8.B
~> $\pi_1(Y, y_0)$ is a free group.

Cov! $\pi_1(q)$ inj.

back to groups: Thus

$$H \cong f^{-1}(H) = \pi_1(q)\big(\pi_1(Y, y_0)\big) \cong \pi_1(Y, y_0) \qquad \square$$

2.4 APPLICATIONS OF π_1 / COVERING THEORY

- undecidability results in top.
- Nielsen-Schreier theorem
- fundamental thm of algebra (Ⓣ 5.4)
- Borsuk-Ulam thm in dim 2 (Ⓣ 8.3)

(
- Brouwer fixed pt thm in dim 2
- Jordan curve thm in dim 2
- classification of compact surfaces

) could be done

$\boxed{3}$ AXIOMATIC HOMOLOGY THEORY

Goal: · introduce the Eilenberg–Steenrod axioms for homology theories
· and computations from these axioms
(Existence: in Chapter $\boxed{4}$).

⇝ we will see which properties of homology theories are "generic".

Basic idea: homotopy invariant + glueing

$\boxed{3.1}$ THE EILENBERG–STEENROD AXIOMS

$\boxed{3.1.1}$ THE AXIOMS

Definition. (Eilenberg–Steenrod axioms for homology theories).
Let R be a ring with unit. A <u>homology theory on Top^2 with values in ${}_R\mathrm{Mod}$</u> is a pair $((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}})$ consisting of:

cat. of left R-modules

data · functors $h_k : \mathrm{Top}^2 \longrightarrow {}_R\mathrm{Mod}$

"replacement for quot."

- natural transformations $\boxed{\partial_k : h_k \Rightarrow h_{k-1} \circ U}$,
  where $U : Top^2 \to Top^2$ is the subspace functor:
  - obj: $(X, A) \mapsto (A, \emptyset)$
  - mor: $(f : (X,A) \to (Y,B)) \mapsto f|_A$

props: and satisfying the following conditions:

- homotopy invariance: for each $k \in \mathbb{Z}$, the functor $h_k : Top^2 \to {}_R\mathrm{Mod}$ is homotopy invariant $(f \underset{A,B}{\simeq} g \Rightarrow h_k(f) = h_k(g))$
- long exact sequence of pairs: For every pair (X,A) of spaces, this sequence is exact:

$$\xrightarrow{\partial_{k+1}} h_k(A,\emptyset) \xrightarrow{h_k(i)} h_k(X,\emptyset) \xrightarrow{h_k(j)} h_k(X,A) \xrightarrow{\partial_k} h_{k-1}(A,\emptyset) \to$$

∂_k for (X,A)

where $i : (A,\emptyset) \hookrightarrow (X,\emptyset)$, $j : (X,\emptyset) \hookrightarrow (X,A)$ are the inclusions

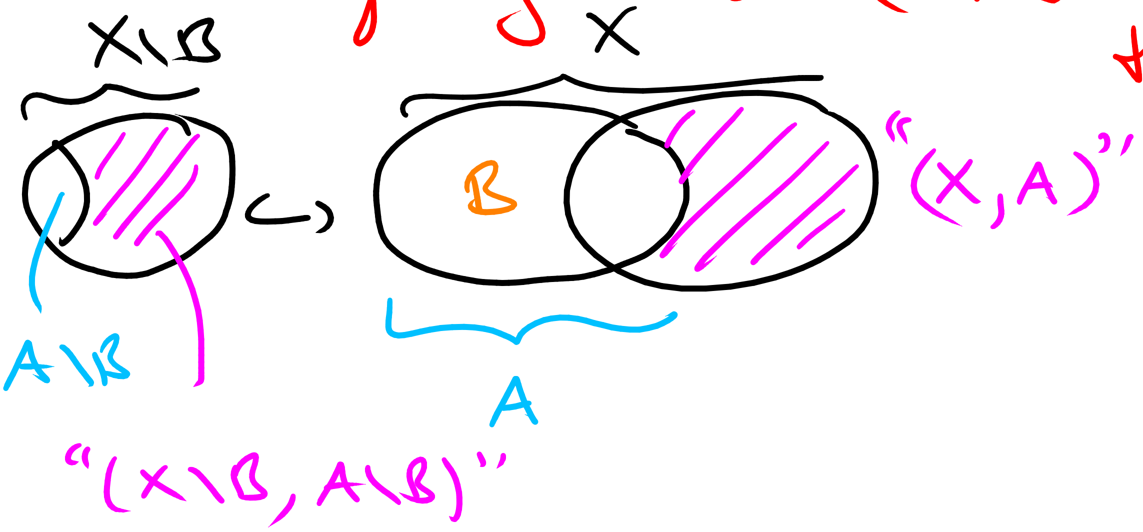
Ausschneidung

- excision: For every pair (X,A) of spaces and all subspaces $B \subset A$ with $\overline{B} \subset \mathring{A}$, the incl. ind. isos $\forall_{k \in \mathbb{Z}}$ $h_k(X \setminus B, A \setminus B) \xrightarrow{\cong} h_k(X, A)$

$B \subset A \subset X$

"$(X-B)-(A-B) = X-A$"

$\rightsquigarrow$ glueing results ($\rightsquigarrow$ can compute things!)



- We write $\bullet := \{\emptyset\}$ for "the" one-pt space. We say that $(h_k(\bullet, \emptyset))_{k \in \mathbb{Z}}$ are the <u>coefficients</u> of the theory. gewöhnlich

- A homology theory (h_*, ∂_*) is <u>ordinary</u> if the dimension axiom is satisfied, i.e.:

$$\forall_{k \in \mathbb{Z} \setminus \{0\}} \quad h_k(\bullet, \emptyset) \cong_R 0.$$

"dim $\bullet$"

- A homology theory (h_*, ∂_*) is <u>additive</u> if the additivity axiom is satisfied, i.e., for all sets I and all families $(X_i)_{i \in I}$ of spaces, the can. incl. $(X_i \to \coprod_{j \in I} X_j)_{i \in I}$ induce isos for all $k \in \mathbb{Z}$:

$$\bigoplus_{i \in I} h_k(X_i, \emptyset) \longrightarrow h_k\Big(\coprod_{i \in I} X_i, \emptyset\Big)$$

Notation: $h_k(X) := h_k(X, \emptyset)$. "absolute homology"

3.1.2 FIRST STEPS

Setup: In the following: R is a ring with unit and $((h_k)_{k\in\mathbb{Z}}, (\partial_k)_{k\in\mathbb{Z}})$ is a homology theory on Top^2 with values in ${}_R\mathrm{Mod}$.

Proposition. (more on homotopy invariance) Let $k \in \mathbb{Z}$.

1. If (X, A) is a pair of spaces and if the incl. $i: A \hookrightarrow X$ is a homotopy equivalence, then

$$h_k(X, A) \cong_R 0.$$

In particular: $h_k(X, X) \cong_R 0$.

Proof of 1: By the LES of (X, A), the following is exact:

$j: (X, \emptyset) \hookrightarrow (X, A)$

$$h_k(A) \xrightarrow[\cong]{h_k(i)} h_k(X) \xrightarrow[0]{h_k(j)} \boxed{h_k(X, A)} \xrightarrow[0]{\partial_k} h_{k-1}(A) \xrightarrow[\cong]{h_k(i)} h_{k-1}(X)$$

(htpy inv!)

exactness

$\rightsquigarrow$ • $\operatorname{im} \partial_2 = \ker h_2(i) = 0$ $\rightsquigarrow \partial_2 = 0$

• $\ker h_2(j) = \operatorname{im} h_2(i) = h_2(X)$

$\rightsquigarrow h_2(j) = 0$

exactness at $h_2(X,A)$

$\rightsquigarrow$ $h_2(X,A) \cong 0.$ $\square$ 1.