Org: lecture on Jan 7, 2022: online (!)

21.12.2021

Recap: axioms for homology theories $((\underset{\sim}{h_k})_{k\in\mathbb{Z}}, (\underset{\sim}{\partial_k})_{k\in\mathbb{Z}})$

$\mathrm{Top}^2 \to {}_R\mathrm{Mod} \qquad h_k \Rightarrow h_{k-1}\circ U$

- homotopy invariance
- LES for pairs
- excision

(• dim axiom
• additivity)

Proposition ...

2. If $f \in \mathrm{map}((X,A),(Y,B))$ such that $f\colon X \to Y$ is a homotopy equivalence and $f|_A\colon A \to B$ is a homotopy eq. (in Top), then: for all $k \in \mathbb{Z}$,
$$h_k(f)\colon h_k(X,A) \longrightarrow h_k(Y,B)$$
is an R-iso.

Proof. 2. We consider the LES for (X,A) and (Y,B):

ind. by ind.

$$\begin{tikzcd}
\text{exact} & h_k(A) \arrow[r] \arrow[d, "h_k(f|_A)", "\cong"'] \arrow[dr, phantom, "\circledast"] & h_k(X) \arrow[r] \arrow[d, "h_k(f)", "\cong"'] \arrow[dr, phantom, "\circledast"] & h_k(X,A) \arrow[r, "\partial_k^{(X,A)}"] \arrow[d, "h_k(f)"] \arrow[dr, phantom, "\circledast\circledast"] & h_{k-1}(A) \arrow[r] \arrow[d, "h_{k-1}(f|_A)", "\cong"'] \arrow[dr, phantom, "\circledast"] & h_{k-1}(X) \arrow[d, "h_{k-1}(f)", "\cong"'] \\
\text{exact} & h_k(B) \arrow[r] & h_k(Y) \arrow[r] & h_k(Y,B) \arrow[r, "\partial_k^{(Y,B)}"'] & h_{k-1}(B) \arrow[r] & h_{k-1}(Y)
\end{tikzcd}$$

$\circledast$ is commutative: h_k, h_{k-1} is a functor

$\circledast\circledast$ is commutative: ∂_k is a nat trafo

The outer vertical arrows are R-isos, because $f: X \to Y$, $f|_A : A \to B$ are htpy equivs and h_k, h_{k-1} are htpy inv.

From the five lemma, we obtain that $h_k(f): h_k(X, A) \to h_k(Y, B)$ is an R-iso. $\square$

Example: let $n \in \mathbb{N}$, let $k \in \mathbb{Z}$. Then the incl. induces an R-iso

$$h_k(D^{n+1}, S^n) \underset{R}{\cong} h_k(\mathbb{R}^{n+1}, \mathbb{R}^{n+1} \setminus \{0\}).$$

Proposition. (splitting off the coefficients).

1. Let (X, x_0) be a ptd space. Then the constant map $X \xrightarrow{p} \{x_0\}$ and the incl. $(X, \emptyset) \overset{j}{\hookrightarrow} (X, \{x_0\})$ induce R-isos
$$h_k(X) \longrightarrow h_k(\{x_0\}) \oplus h_k(X, \{x_0\})$$
for all $k \in \mathbb{Z}$.
2. This is natural: If $f \in \mathrm{map}_*((X, x_0), (Y, y_0))$ then the diag is commutative:

$$\begin{array}{ccc} h_k(X) & \xrightarrow[1.]{\cong_R} & h_k(\{x_0\}) \oplus h_k(X, \{x_0\}) \\ {\scriptstyle h_k(f)} \downarrow & & \downarrow {\scriptstyle h_k(f|_{\{x_0\}}) \oplus h_k(f)} \\ h_k(Y) & \xrightarrow[\cong_R]{1.} & h_k(\{y_0\}) \oplus h_k(Y, \{y_0\}) \end{array}$$

Proof. 2. functoriality of h_ξ.

1. We start with the LES of $(X, \{x_0\})$:

$$\begin{tikzcd} h_\xi(\{x_0\}) \arrow[r, "h_\xi(i)"] & h_\xi(X) \arrow[l, bend left, dashed, "h_\xi(p)"] \arrow[r, "h_\xi(j)"] & h_\xi(X, \{x_0\}) \arrow[r, "\partial_\xi"] & h_{\xi-1}(\{x_0\}) \arrow[d, "h_{\xi-1}(i)"] \\ & & & h_{\xi-1}(X) \end{tikzcd}$$

From $p \circ i = \mathrm{id}_{\{x_0\}}$, we obtain

$h_\xi(p) \circ h_\xi(i) = \mathrm{id}_{h_\xi(\{x_0\})}$ (same for $\xi - 1$).

In particular: $h_\xi(i)$, $h_{\xi-1}(i)$ are inj.

$\overset{\text{LES}}{\rightsquigarrow}$ $\partial_\xi = 0$.

$\overset{\text{LES}}{\rightsquigarrow}$ we obtain a short exact seq:

$$\begin{tikzcd} 0 \arrow[r] & h_\xi(\{x_0\}) \arrow[r, "h_\xi(i)"] & h_\xi(X) \arrow[l, bend left, dashed, "h_\xi(p)"] \arrow[r, "h_\xi(j)"] & h_\xi(X, \{x_0\}) \arrow[r, "0"] & 0 \end{tikzcd}$$

This SES splits via $h_\xi(p)$.

$\rightsquigarrow$ claim.

Prop. A.6.1. $\square$

Remark. (reduced homology). If X is a top. space and $k \in \mathbb{Z}$, then the <u>reduced homology of X</u> in degree k (wrt (h_*, ∂_*)) is

$X \to \bullet$ constant map

$$\tilde{h}_k(X) := \ker\left(h_k(c_X): h_k(X) \longrightarrow h_k(\bullet)\right) \subset h_k(X).$$

Then: for any $x_0 \in X$, the composition

$$\boxed{\tilde{h}_k(X)} \hookrightarrow h_k(X) \xrightarrow{h_k(\mathrm{incl})} \boxed{h_k(X, \{x_0\})}$$

is an R-iso. ($\tilde{h}$) 10.2

(This is functorial).

$(A,B) \to (X,B) \xrightarrow{h} (X,A)$

Proposition. (LES for triples). Let X be a top. space and let $B \subset A \subset X$ be subspaces. Then the following seq of R-modules is exact:

$$\to h_k(A,B) \xrightarrow{h_k(\mathrm{incl.})} h_k(X,B) \xrightarrow{h_k(\mathrm{incl.})} h_k(X,A) \xrightarrow{\partial_k^{(X,A,B)}} h_{k-1}(A,B)$$

where $\partial_k^{(X,A,B)}$ is def'd by the comp.

$$\begin{tikzcd} h_k(X,A) \arrow[rr, dashed] \arrow[dr, "\partial_k^{(X,A)}"'] & & h_{k-1}(A,B) \\ & h_{k-1}(A) \arrow[ur, "h_{k-1}(\mathrm{incl.})"'] & \end{tikzcd}$$

Sketch of proof: We weave a braid from the LES for (A,B), (X,B), (X,A):

$$
\begin{tikzcd}[column sep=small]
h_{k+1}(X,A) \arrow[dr, "\partial_{k+1}"'] \arrow[rr, bend left, "\partial_{k+1}^{(X,A,B)}"] & & h_k(A,B) \arrow[dr] \arrow[rr, bend left, "\partial_k"] \arrow[ddr, dashed] & & h_{k-1}(B) \arrow[dr] & & h_{k-1}(X) \\
& h_k(A) \arrow[ur] \arrow[dr] & & h_k(X,B) \arrow[ur, "\partial_k"] \arrow[dr] & & h_{k-1}(A) \arrow[ur] \arrow[dr] & \\
h_k(B) \arrow[ur] \arrow[rr, bend right] & & h_k(X) \arrow[ur] \arrow[rr, bend right] & h_k(A,A) \arrow[r, dashed] & h_k(X,A) \arrow[ur, "\partial_k"] \arrow[rr, bend right, "\partial_k^{(X,A,B)}"'] & & {}
\end{tikzcd}
$$

- This is commutative (check!)
- The comp. of subsequent red arrows is zero:
  - at $h_k(X,B)$: factors over $\underbrace{h_k(A,A)}_{\cong_R 0}$.
  - at $h_k(A,B)$: factors over $\underbrace{h_{k+1}(X,A) \xrightarrow{\partial_{k+1}} h_k(A) \to h_k(X)}_{=0 \text{ (LES)}}$
  - at $h_k(X,A)$: as for $h_k(A,B)$.

. Rest of exactness: diagram chase (check!) □

3.2 HOMOLOGY OF SPHERES AND SUSPENSIONS

Goal: compute homology of spheres

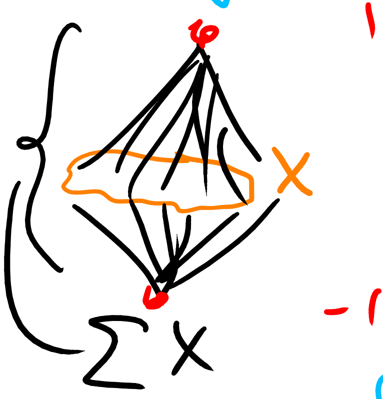
idea: inductively, over suspensions

Setup. Let R be a ring with unit and let (h_*, ∂_*) be a homology thy on Top^2 with values in ${}_R Mod$.

3.2.1 SUSPENSIONS

Recall: The (unreduced) suspension functor $\Sigma : Top \to Top$ is def'd by

. obj: $\Sigma X := X \times [-1,1] / \forall x, x' \in X \; (x,1) \sim (x',1), \; (x,-1) \sim (x',-1)$

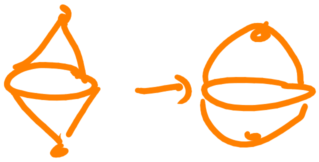


. morphisms:

$f \in map(X,Y) \rightsquigarrow \Sigma f : \Sigma X \to \Sigma Y$
$[x,t] \mapsto [f(x),t]$

(h) 2.2

Example. For $n \in \mathbb{N}$, we have a homeo (û) 2.2



$$\Sigma S^n \longrightarrow S^{n+1}$$

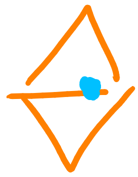
$$[x,t] \longmapsto \left(x\cos\left(\tfrac{\pi}{2}t\right), \sin\left(\tfrac{\pi}{2}t\right)\right)$$

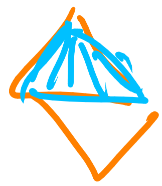
Theorem. (homology of suspensions). Let (X, x_0) be a ptd space, $k \in \mathbb{Z}$. Then there is a natural (on Top_*) R-iso (<u>suspension iso</u>):

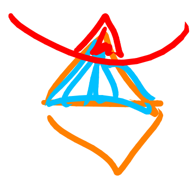
$$\sigma_k(X,x_0) : h_k(X, \{x_0\}) \longrightarrow h_{k+1}(\Sigma X, \{[x_0, 0]\}).$$

Proof. We construct $\sigma_k(X,x_0)$ as a composition of natural isos.

$$\begin{array}{ll}
 & h_{k+1}(\Sigma X, \{[x_0,0]\}) \\
\text{ind by ind.} & \big\downarrow \cong_R \quad (\text{Prop. } 3.1.4) \\
 & h_{k+1}(\Sigma X, C_+X) \\
\text{ind. by ind.} & \big\uparrow \cong_R \quad (\text{exc.}) \\
 & h_{k+1}(\Sigma X \setminus \tfrac{1}{2}C_+X, C_+X \setminus \tfrac{1}{2}C_+X) \\
\text{ind by ind} & \big\uparrow \cong_R \quad (\text{Prop. } 3.1.4) \\
 & h_{k+1}(C_-X, X \times \cancel{\{0\}})
\end{array}$$







$$\partial^{\text{triple}}_{k+1} \qquad h_k(X, \{x_0\})$$

$*$ is an iso: triple LES:

$$\underbrace{h_{k+1}(C_-X, \{x_0\})}_{\cong_R 0} \to h_{k+1}(C_-X, X) \xrightarrow{\partial^{\text{triple}}_{k+1}} h_k(X, \{x_0\}) \to \underbrace{h_k(C_-X, \{x_0\})}_{\cong_R 0}$$

because $C_-X \simeq \bullet$

(L) 3.4

$$\cong_R$$

□