

Algebraic Topology I: Exercises

Prof. Dr. C. Löh/F. Hofmann

Sheet 1, October 15, 2026

Hinweis. The Études will not be submitted/graded. You may use them as warm-up exercises in your daily practice. Some of these problems will also be discussed in the exercise classes.

Étude A (Recap: point-set topology). Recall the following notions: topological space, continuous map, topology induced by a metric, compact space.

Hint. For us, compactness does *not* include the Hausdorff property.

Étude B (quotient spaces). Sketch the following quotient spaces:

$$[0, 1]/\{0, 1\}, \quad [0, 1]/\{0, 1/2\}, \quad [0, 1]/\{0, 1/2, 1\}, \quad [0, 1]/[0, 1/2]$$

Étude C (an exotic quotient). We consider $X := [0, 1]/(\mathbb{Q} \cap [0, 1])$, endowed with the quotient topology of the standard topology on $[0, 1]$.

1. Is X compact?
2. Is X Hausdorff?
3. Is X path-connected?
4. Is X homeomorphic to S^1 ?

Étude D (homeomorphism classification). Classify the following topological spaces up to homeomorphism:

$$\mathbb{R}^2, \quad \mathbb{R}, \quad [0, 1], \quad (0, 1), \quad S^1$$

Hints. Please adhere to the standards outlined in the notes on organisational matters: https://loeh.app.ur.de/teaching/topologie1_ws2627/org.pdf

Exercise 1 (BLORX; 4 Credits). Prove that no two of the following five subsets of $\mathbb{R}^2$ are homeomorphic.

BLORX

Hint. Try to find a single principle that works in all these cases. Probably, the most difficult case is to distinguish B and O.

Exercise 2 (quotient topology; 4 (=2+2) Credits). Let X be a topological space, let $A \subset X$, and let $\pi: X \rightarrow X/A$ be the canonical projection. We equip X/A with the quotient topology. Which of the following statements are in this situation always true? Justify your answer!

1. If $Y \subset X$ is a closed subset, then $\pi(Y) \subset X/A$ is closed.
2. If $Y \subset X/A$ is a closed subset, then there exists a closed subset $\tilde{Y} \subset X$ with $\pi(\tilde{Y}) = Y$.

Please turn over

Exercise 3 (stereographic projection; 4 Credits). Let $n \in \mathbb{N}_{>0}$ and let $N \in S^n$ be the North Pole, i.e., $N := (0, \dots, 0, 1) \in S^n$. The map

$$s_n: S^n \setminus \{N\} \longrightarrow \mathbb{R}^n$$

$$(x_1, \dots, x_{n+1}) \longmapsto \frac{1}{1 - x_{n+1}} \cdot (x_1, \dots, x_n)$$

is called *stereographic projection*. Give a geometric interpretation of this map and prove that it is a homeomorphism. Illustrate your arguments graphically!

Exercise 4 (balls, spheres, simplices; 4 Credits). Let $n \in \mathbb{N}_{>0}$. Solve one of the following problems:

1. Prove that Δ^n is homeomorphic to D^n and that $\partial\Delta^n$ is homeomorphic to S^{n-1} .

Hint. Find the centre and inflate!

2. Prove that D^n/S^{n-1} is homeomorphic to S^n .

Illustrate your arguments graphically!

Hint. The compact-Hausdorff trick might be useful.

Bonus Problem (Peano curves; 4 (=2+2) Credits). Show that there exist surjective continuous maps $[0, 1] \longrightarrow [0, 1] \times [0, 1]$. Can such a map be injective? Justify your answer.