Algebraic Topology I: Exercises

Prof. Dr. C. Löh/F. Hofmann — Sheet 2, October 22, 2026

Étude A (functors on pairs of spaces)**.** Give definitions of functors $\mathsf{Top}^2 \longrightarrow \mathsf{Top}$ that deserve the following names: subspace functor, ambient space functor, quotient space functor

Étude B (morphisms in the simplex category)**.** In the simplex category Δ, describe the following sets of morphisms explicitly and draw schematic pictures of these morphisms:

$$\mathrm{Mor}_\Delta\big(\Delta(1), \Delta(2)\big), \quad \mathrm{Mor}_\Delta\big(\Delta(0), \Delta(2)\big), \quad \mathrm{Mor}_\Delta\big(\Delta(2), \Delta(1)\big)$$

Étude C (pushouts, concretely)**.** Let $* := \{0\}$ denote "the" one-point space. Let $n \in \mathbb{N}$. Determine the pushouts of the following diagrams of topological spaces, where $S^n \longrightarrow *$ is the constant map and $S^n \longrightarrow D^{n+1}$ is the inclusion:

$$\begin{array}{ccc} S^n & \longrightarrow & * \\ \downarrow & & \\ * & & ? \end{array} \qquad \begin{array}{ccc} S^n & \longrightarrow & * \\ \downarrow & & \\ D^{n+1} & & ? \end{array} \qquad \begin{array}{ccc} S^n & \longrightarrow & D^{n+1} \\ \downarrow & & \\ D^{n+1} & & ? \end{array}$$

Étude D (pushouts, abstractly)**.** Let X be a topological space, let $A \subset X$ be a subset (with the subspace topology), let $i\colon A \longrightarrow X$ be the inclusion, and let $c\colon A \longrightarrow *$ be the unique map. Determine the pushout of the following diagram of topological spaces:

$$\begin{array}{ccc} A & \xrightarrow{c} & * \\ {\scriptstyle i}\downarrow & & \\ X & & ? \end{array}$$

Hint. Distinguish two cases, depending on whether A is empty or not.

Hints. Please adhere to the standards outlined in the notes on organisational matters: https://loeh.app.ur.de/teaching/topologie1_ws2627/org.pdf

Exercise 1 (projective spaces via glueings; 4 Credits)**.** Let $n \in \mathbb{N}$. Prove that there is a pushout diagram of topological spaces of the following type:

$$\begin{array}{ccc} S^n & \longrightarrow & \mathbb{R}P^n \\ \downarrow & & \downarrow \\ D^{n+1} & \longrightarrow & \mathbb{R}P^{n+1} \end{array}$$

In particular, describe all of the maps in these diagrams explicitly and illustrate your arguments graphically.

Hint. If you prefer a more concrete setting, you may instead of the general case just consider the two cases of $n = 0$ and $n = 1$.

Exercise 2 (relative topology; 4 (=2+2) Credits)**.** Let (X, A) and (Y, B) be pairs of topological spaces. Which of the following statements are in this situation always true? Justify your answer!

1. If $(X, A) \cong_{\mathsf{Top}^2} (Y, B)$, then $A \cong_{\mathsf{Top}} B$ and $X \cong_{\mathsf{Top}} Y$.
2. If $X/A \cong_{\mathsf{Top}} Y/B$, then $(X, A) \cong_{\mathsf{Top}^2} (Y, B)$.

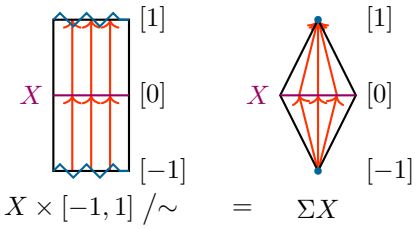
Please turn over

Exercise 3 (suspension/Einhängung; 4 (=2+2) Credits)**.** For a topological space X, we define the *suspension of* X as

$$\Sigma X := \big(X \times [-1,1]\big) \,/\!\sim$$

(endowed with the quotient topology of the product toplogy), where "$\sim$" is the equivalence relation generated by

$$\begin{aligned} &\forall_{x,x' \in X} \quad (x,1) \sim (x',1) \\ &\forall_{x,x' \in X} \quad (x,-1) \sim (x',-1). \end{aligned}$$



1. Let $f\colon X \longrightarrow Y$ be a continuous map. Show that the map Σf is well-defined and continuous:

$$\begin{aligned} \Sigma f\colon \Sigma X &\longrightarrow \Sigma Y \\ [x,t] &\longmapsto \big[f(x),t\big] \end{aligned}$$

2. Let $n \in \mathbb{N}$. Show that the following map is a well-defined homeomorphism:

$$\begin{aligned} \Sigma S^n &\longrightarrow S^{n+1} \\ [x,t] &\longmapsto \big(\cos(\pi/2 \cdot t) \cdot x, \sin(\pi/2 \cdot t)\big) \end{aligned}$$

Exercise 4 (morphisms in the simplex category; 4 (=2+2) Credits)**.** For $n \in \mathbb{N}_{>0}$ and $j \in \{0,\dots,n\}$ we define

$$\begin{aligned} d^n_j\colon \Delta(n-1) &\longrightarrow \Delta(n) \\ k &\longmapsto \begin{cases} k & \text{if } k < j \\ k+1 & \text{if } k \geq j; \end{cases} \end{aligned}$$

for $n \in \mathbb{N}$ and $j \in \{0,\dots,n\}$ we define

$$\begin{aligned} s^n_j\colon \Delta(n+1) &\longrightarrow \Delta(n) \\ k &\longmapsto \begin{cases} k & \text{if } k \leq j \\ k-1 & \text{if } k > j. \end{cases} \end{aligned}$$

Clearly, all of these maps are morphisms in the simplex category Δ.

1. Prove that every morphism in Δ is a composition of finitely many of the morphisms above.

2. Let $n \in \mathbb{N}$. Prove that for all $j,k \in \{0,\dots,n+1\}$ with $j < k$ we have

$$d^{n+1}_k \circ d^n_j = d^{n+1}_j \circ d^n_{k-1}.$$

Bonus Problem (torus games; 4 (=2+2) Credits)**.** We use the description of the torus as quotient of a square (Example 1.1.16).

1. Would it be reasonable to play chess or Tic-Tac-Toe on a torus (as quotient of the square-shaped board)? Justify your claim!

2. Name two computer games (except Asteroids an its clones) that are played on a torus under such an identification. Don't forget to cite sources appropriately and justify your claim!

Submission by October 29, 2026, 8:20, via mailbox/GRIPS