Algebraic Topology I: Exercises

Prof. Dr. C. Löh/F. Hofmann — Sheet 3, October 29, 2026

Étude A (homotopy)**.** Which of the following statements are true for all topological spaces X and all continuous maps $f\colon X \longrightarrow X$?

1. If $f \simeq \mathrm{id}_X$, then f is bijective.
2. If $f \simeq \mathrm{id}_X$, then $f \circ f \simeq f$.
3. If $f \circ f \simeq f$, then $f \simeq \mathrm{id}_X$.
4. If $f^{2026} \simeq \mathrm{id}_X$, then f is a homotopy equivalence.

Étude B (homotopy classes)**.** Describe for the following types of maps all possible homotopy classes:

$$S^0 \longrightarrow [0,1], \qquad S^0 \longrightarrow S^0, \qquad [0,1] \longrightarrow S^0, \qquad \mathbb{R}^{2026} \longrightarrow \mathbb{R}^{42}$$

Étude C ($\mathbb{R}$-homotopy)**.** Let X and Y be topological spaces. Continuous maps $f, g\colon X \longrightarrow Y$ are *$\mathbb{R}$-homotopic* if there exists a continuous map $h\colon X \times \mathbb{R} \longrightarrow Y$ with $h(\,\cdot\,,0) = f$ and $h(\,\cdot\,,1) = g$. How are the notions of "homotopic" and "$\mathbb{R}$-homotopic" related?

Étude D (homotopy invariant functors)**.** Which of the following statements are true for all homotopy invariant functors $F\colon \mathsf{Top} \longrightarrow \mathsf{Ab}$?

1. $F(D^{2026}) \cong_{\mathsf{Ab}} F(\mathbb{R}^{2027})$.
2. $F(S^{2026}) \cong_{\mathsf{Ab}} \mathbb{Z}^{2026}$.
3. $F(\mathbb{R}P^{2026}) \cong_{\mathsf{Ab}} F(\mathbb{R} \times \mathbb{R}P^{2026})$.
4. If X is contractible, then $F(X) \cong_{\mathsf{Ab}} \{0\}$.

Hint. All of these problems are easy!

Hints. Please adhere to the standards outlined in the notes on organisational matters: https://loeh.app.ur.de/teaching/topologie1_ws2627/org.pdf

Exercise 1 (homotopy equivalence and path-connectedness; 4 (=2+2) Credits)**.**

1. Show that a space X is path-connected if and only if all continuous maps $[0,1] \longrightarrow X$ are homotopic to each other.
2. Conclude that path-connectedness is preserved under homotopy equivalences.

Exercise 2 (point removal trick for homotopy equivalence? 4 (=2+2) Credits)**.** Let X and Y be topological spaces and let $x \in X$, $y \in Y$. Which of the following statements are in this situation always true? Justify your answer!

1. If $X \simeq Y$, then $X \setminus \{x\} \simeq Y \setminus \{y\}$.
2. If $X \setminus \{x\} \simeq Y \setminus \{y\}$, then $X \simeq Y$.

Please turn over

Exercise 3 (OOOOOO; 4 Credits)**.** We consider the folowing three subspaces of $\mathbb{R}^2$. State and prove the classification of these spaces up to homotopy equivalence.



Exercise 4 (mapping cones/Abbildungskegel; 4 (=2+2) Credits)**.** Let X and Y be topological spaces and let $f \in \mathrm{map}(X, Y)$. The *mapping cone of* f is defined by the pushout

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i}\downarrow & & \downarrow \\ \mathrm{Cone}(X) & \longrightarrow & \mathrm{Cone}(f) \end{array}$$

where $\mathrm{Cone}(X) := (X \times [0,1])/(X \times \{0\})$ denotes the *cone over* X and $i\colon X \longrightarrow \mathrm{Cone}(X)$, $x \longmapsto [x, 1]$ is the canonical map to the base of the cone. Prove the following statements (and illustrate your arguments graphically), provided that X is non-empty:

1. The cone $\mathrm{Cone}(X)$ is contractible.
2. If $f\colon X \longrightarrow Y$ is a homotopy equivalence, then the mapping cone $\mathrm{Cone}(f)$ is contractible.

Bonus Problem (four-point circle; 4 (=2+2) Credits)**.** Let $X := \{1, 2, 3, 4\}$. We equip X with the topology generated by $\big\{\{1\}, \{3\}, \{1, 2, 3\}, \{1, 3, 4\}\big\}$. In this exercise, you may use that S^1 is *not* contractible (which we will prove later in this course).

1. Show that there exists a surjective continuous map $S^1 \longrightarrow X$.
2. Show that S^1 and X are *not* homotopy equivalent.
3. *Bonus bonus problem.* Show that X is *not* contractible.

Submission by November 5, 2026, 8:20, via mailbox/GRIPS