Algebraic Topology I: Exercises

Prof. Dr. C. Löh/F. Hofmann — Sheet 4, November 5, 2026

Étude A (classification problem)**.** In this exercise, you may assume that the theorem on existence of "interesting" homotopy invariant functors holds. Classify the following spaces up to homeomorphism/homotopy equivalence.

$$\mathbb{R}^{2026}, \qquad \mathbb{R}^{2027}, \qquad S^{2026}, \qquad D^{2027}, \qquad S^0 \times S^{2026}, \qquad S^0 \times S^{2027}$$

Étude B (concatenation of loops)**.** Let $\gamma := \mathrm{id}_{S^1} \colon (S^1, e_1) \longrightarrow (S^1, e_1)$. Draw pictures of the following loops:

$$\overline{\gamma}, \qquad \gamma * \gamma, \qquad \gamma * (\gamma * \gamma), \qquad \gamma * \overline{\gamma}, \qquad (\gamma * \overline{\gamma}) * \gamma$$

Étude C (the equator)**.** Let $\gamma \colon (S^1, e_1) \longrightarrow (S^2, e_1)$ be the inclusion as equator. Show that $[\gamma]_*$ is trivial in $\pi_1(S^2, e_1)$.

Étude D (induced homomorphisms on π_1)**.** Illustrate the effect on π_1 of the homomorphisms induced by the following maps $(S^1, e_1) \longrightarrow (S^1, e_1)$:

1. $[t] \longmapsto [t]$
2. $[t] \longmapsto [2 \cdot t \mod 1]$
3. $[t] \longmapsto [-t \mod 1]$
4. $[t] \longmapsto [t^2 \mod 1]$

Hint. You may use that the following map is a group isomorphism:

$$\begin{aligned} \mathbb{Z} &\longrightarrow \pi_1(S^1, e_1) \\ d &\longmapsto \big[[t] \mapsto [d \cdot t \mod 1]\big]_* \end{aligned}$$

Hints. Please adhere to the standards outlined in the notes on organisational matters: https://loeh.app.ur.de/teaching/topologie1_ws2627/org.pdf

Exercise 1 (the Möbius strip; 4 Credits)**.** In this exercise, you may assume that the theorem on existence of "interesting" homotopy invariant functors holds. Solve one of the following:

1. Let $n \in \mathbb{N}_{>0}$ and let $H^n := \{x \in \mathbb{R}^n \mid x_n \geq 0\}$ be the upper half-space (with respect to the subspace topology of $\mathbb{R}^n$). Prove *invariance of the boundary*, i.e., show that there is no open neighbourhood of 0 in H^n that is homeomorphic to the open unit ball $(D^n)^\circ$ in $\mathbb{R}^n$.

   Hint. As first step, prove the following: If $U \subset H^n$ is an open neighbourhood of 0 in H^n, then $U \setminus \{0\} \simeq U$.

2. Show that the Möbius strip is *not* homeomorphic to $S^1 \times [0, 1]$.

   Hint. Use invariance of the boundary and strips of paper.

Please turn over

Exercise 2 (spheres and π_n; 4 (=2+2) Credits)**.** Let $n \in \mathbb{N}$. Which of the following statements are in this situation always true? Justify your answer!

1. If $\pi_n(S^n, e_1)$ is infinite, then $\pi_n(\mathbb{R}^n \setminus \{0, e_1\}, -e_1)$ is infinite.
2. If $\pi_n(S^n, e_1)$ consists of a single element, then for every pointed topological space (X, x_0) also $\pi_n(X, x_0)$ consists of a single element.

Exercise 3 (π_0 and path-connected components; 4 Credits)**.** Look up the definition of path-connected components of topological spaces (don't forget to cite the used literature appropriately!). Let (X, x_0) be a pointed space and let $\mathrm{PC}(X)$ be the set of path-connected components of X. Prove that the following map is a well-defined bijection:

$$\begin{aligned} \pi_0(X, x_0) &\longrightarrow \mathrm{PC}(X) \\ [\gamma]_* &\longmapsto [\gamma(-1)] \end{aligned}$$

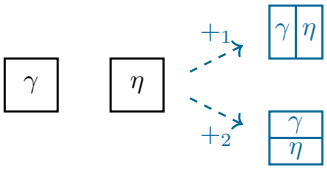
Exercise 4 (π_1 and contractibility; 4 (=2+2) Credits)**.** Let X be a topological space.

1. Let $\gamma\colon S^1 \longrightarrow X$ be a null-homotopic map and let $x_0 := \gamma(1)$. Show that then $[\gamma]_*$ is trivial in $\pi_1(X, x_0)$. Illustrate your argument in a suitable way!
2. Conclude: If X is contractible (but not necessarily pointedly contractible!) and $x_0 \in X$, then $\pi_1(X, x_0)$ is the trivial group.

Bonus Problem (group strcuture on π_n; 4 (=2+2) Credits)**.** Let $n \in \mathbb{N}_{\geq 2}$ and let the composition maps $+_1, \ldots, +_n\colon \pi_n(X, x_0) \times \pi_n(X, x_0) \longrightarrow \pi_n(X, x_0)$ be defined as in Outlook 2.1.6.

1. Prove that $+_j = +_1$ for all $j \in \{1, \ldots, n\}$.
2. Prove that $\pi_n(X, x_0)$ is an Abelian group with respect to $+_1$.

Hint. Eckmann and Hilton might help!



Submission by November 12, 2026, 8:20, via mailbox/GRIPS